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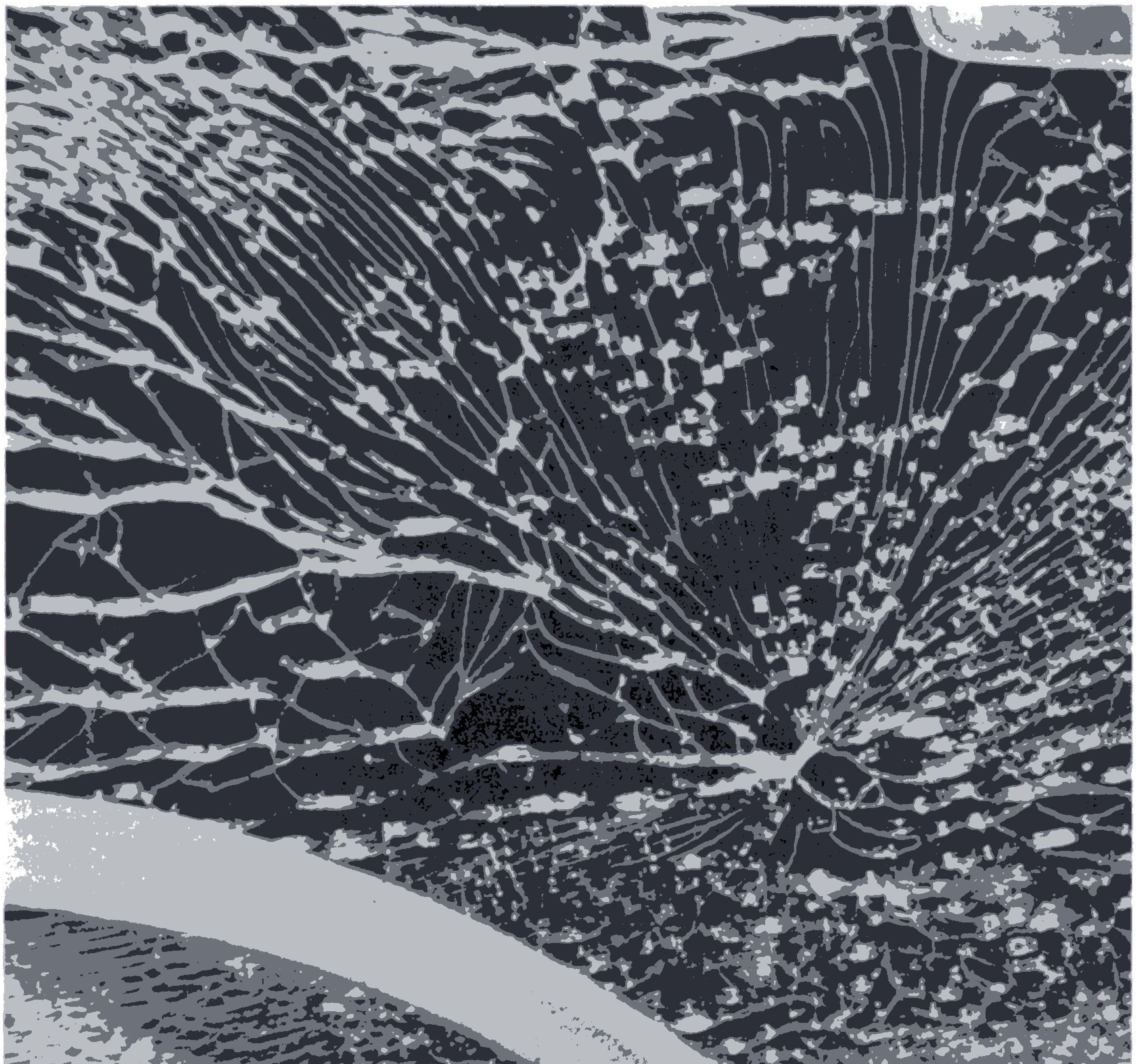

Shock Measurement

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Introduction to

Measurement and Description of Shock

by

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ABSTRACT

Electronic measurement of shock phenomena is becoming more and more commonplace. This article reviews some basic concepts in shock measurement and analysis, and attempts to define suitable criteria for the accurate measurement and recording of shock waveforms.

SOMMAIRE

La mesure électronique des phénomènes de choc devient de jour en jour plus courante et prend dans le domaine de la recherche une place toujours plus importante. Cet article rappelle quelques idées fondamentales de la conception de mesures et d'analyse de choc. Il tente a definir quelques critéres utiles pour la mesure précise et l'enregistrement de la forme d'onde.

ZUSAMMENFASSUNG

Die elektronische Messung von Stoßvorgängen wird immer mehr zu einer alltäglichen Angelegenheit. Dieser Artikel beleuchtet einige Grundprinzipien der Stoßmessung und -Analyse und versucht, geeignete Kriterien für die genaue Messung und Aufzeichung der Kurvenformen von stössen zu definieren.

Introduction

Shock phenomena are encountered relatively often in measurement engineering. They originate from explosions, impacts, earthquakes, supersonic motion and other spasmodic releases of energy. It is a little difficult to define exactly what a shock is, since any type of motion which is not purely periodic, may be thought of as a shock motion, but the following definition is adequate for the present purposes:

A shock is a transmission of (kinetic) energy to a system, which takes place in a relatively short time compared with the natural period of the system, and which is followed by a natural decay of the (oscillatory) motion given to the system.

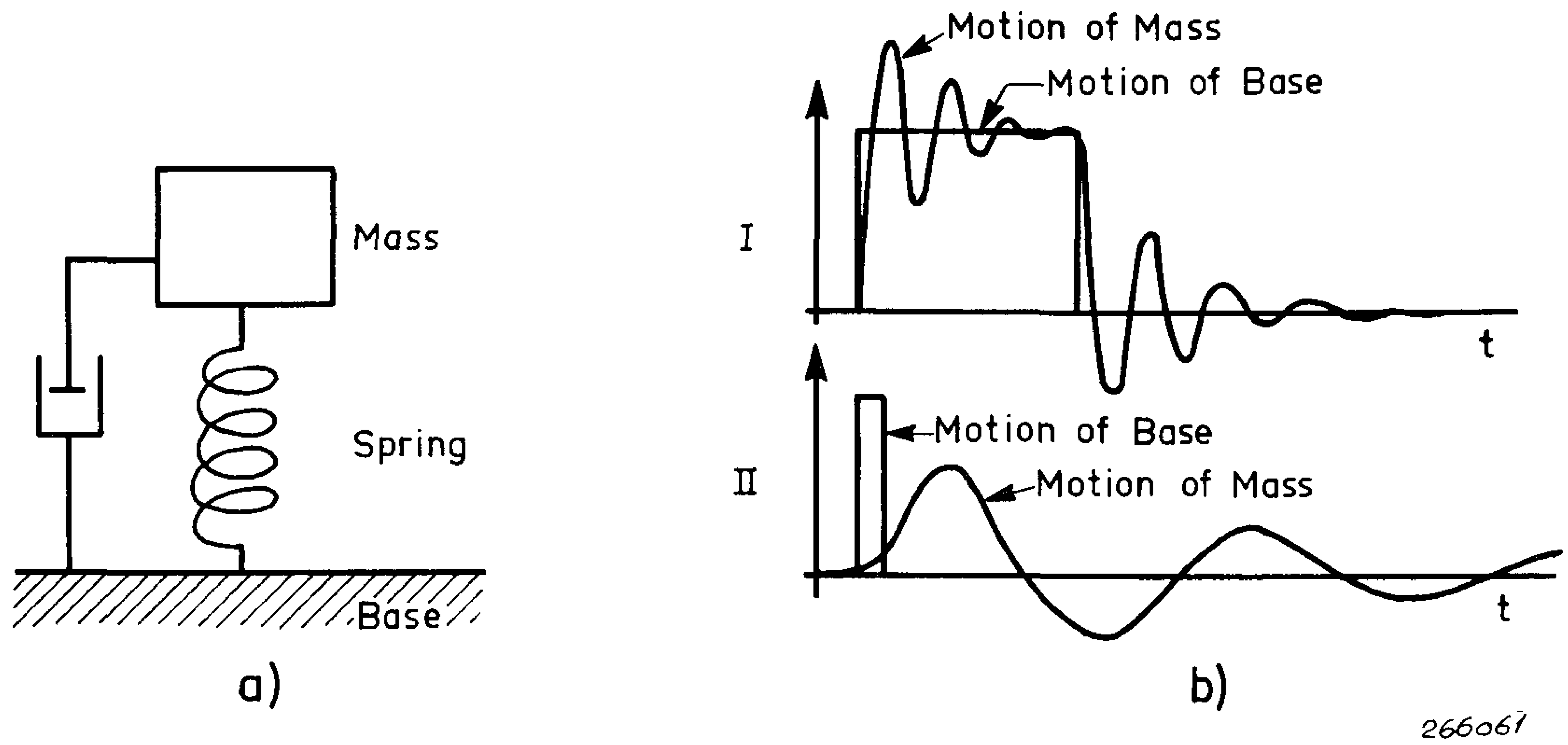


Fig. 1. Response of simple oscillator to a shock input.

A square wave may therefore constitute one or two shocks, depending upon the natural period of the system influenced by it. Fig. 1 illustrates this. The input may be the motion of the base of a spring mass system as shown in Fig. 1a, and the output may be the motion of the mass. In case I the natural period of the system is short compared with the length of the input pulse and the system therefore experiences two shocks (two step functions, one in each direction) with subsequent decay of the motion. In case II the natural period is much longer than the pulse width and the resulting motion is that of a single shock. (Impulse excitation).
The recording and measurement of shock waveforms or transients impose rather stringent requirements upon the instrumentation employed.
It is the purpose of this article to call attention to these requiremens and to review the most common methods used for measuring and describing shock waveforms. Mechanical shock is referred to throughout but the results obtained apply equally well to other shock phenomena.

Description of a Shock

A shock may be measured in terms of acceleration, velocity or displacement and for a complete description it is necessary to give an exact amplitude versus time history of the quantity in question.
In many cases the ultimate goal is not the waveform itself, but rather the effect that the corresponding shock would have on a certain mechanical system. Some sort of waveform analysis is then required, in much the same way as a power spectrum is used to describe continuous waveforms. Such information is helpful in the design of equipment which is to be subjected to shock environments, either in service or in production and transport stages.

Fourier Spectrum

One method of description often used is the counterpart of the frequency spectrum in continuous waveform analysis. It is called the Fourier Spectrum of the shock wave. Such a spectrum gives the distribution of energy contained in the shock wave over the frequency range zero to infinity (or from minus infinity to plus infinity).

The mathematical derivation of the Fourier spectrum is found in many textbooks and in the Appendix to this article. Different authors give slightly different expressions depending upon whether they use ω or f as a variable and whether they consider frequencies from 0 to ∞ or from $-\infty$ to $+\infty$. The differences are of course only formal.

The matematical expression for the Fourier spectrum is

$$F(f) = \int_{-\infty}^{\infty} F(t)\, e^{-j2\pi ft}\, dt$$

where F(t) is the amplitude time history of the shock wave and f is the frequency under consideration.

Solving this for a single rectangular pulse of amplitude A and length T gives the following expression

$$F(f) = \int_{-T/2}^{T/2} A\, e^{-j2\pi ft}\, dt$$

or

$$\left| F(f) \right| = AT \left| \frac{\sin \pi f t}{\pi f t} \right|$$

Similar calculation for a final peak sawtooth pulse gives

$$\left| F(f) \right| = \frac{AT}{2} \left| \frac{1}{\pi f T} \sqrt{1 - \frac{1}{\pi f T} \sin 2\pi f T + \left(\frac{1}{\pi f T} \sin \pi f t \right)} \right|$$

and for a half sine pulse

$$\left| F(f) \right| = \frac{2AT}{\pi} \left| \frac{\cos \pi f T}{1 - 4 f^2 T^2} \right|$$

These functions are plotted in Figs. 2, 3 and 4 for the frequency range $0 \angle f \angle 3/T$.

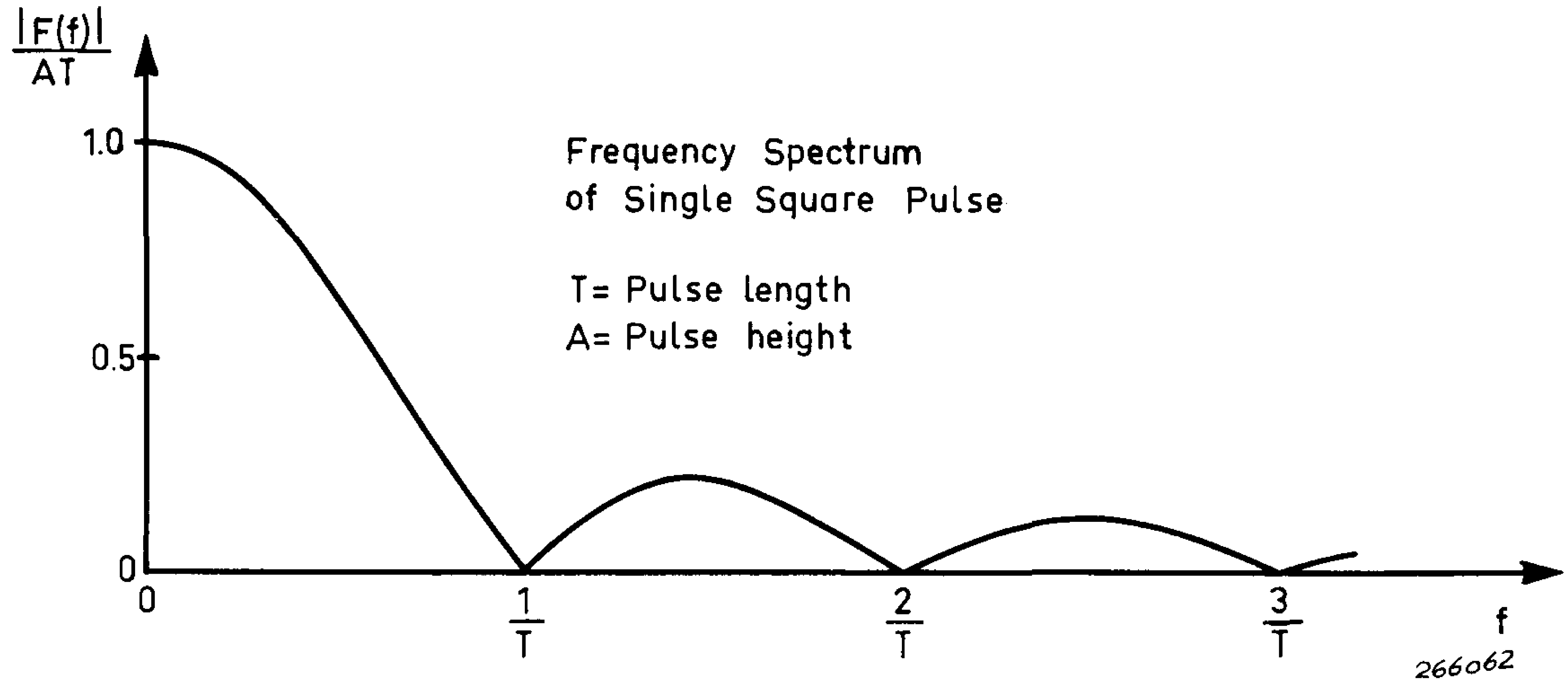


Fig. 2. Frequency spectrum of rectangular pulse.

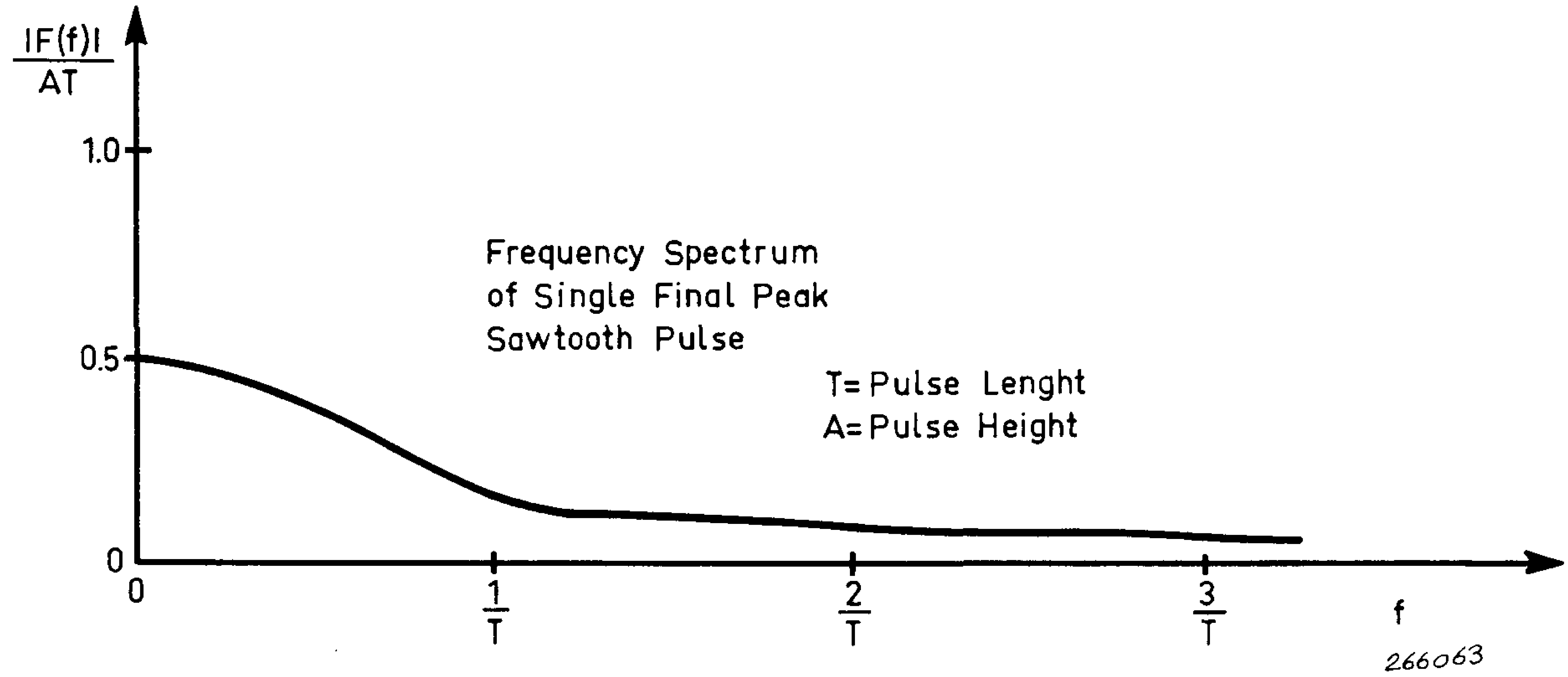


Fig. 3. Frequency spectrum of final peak sawtooth pulse.

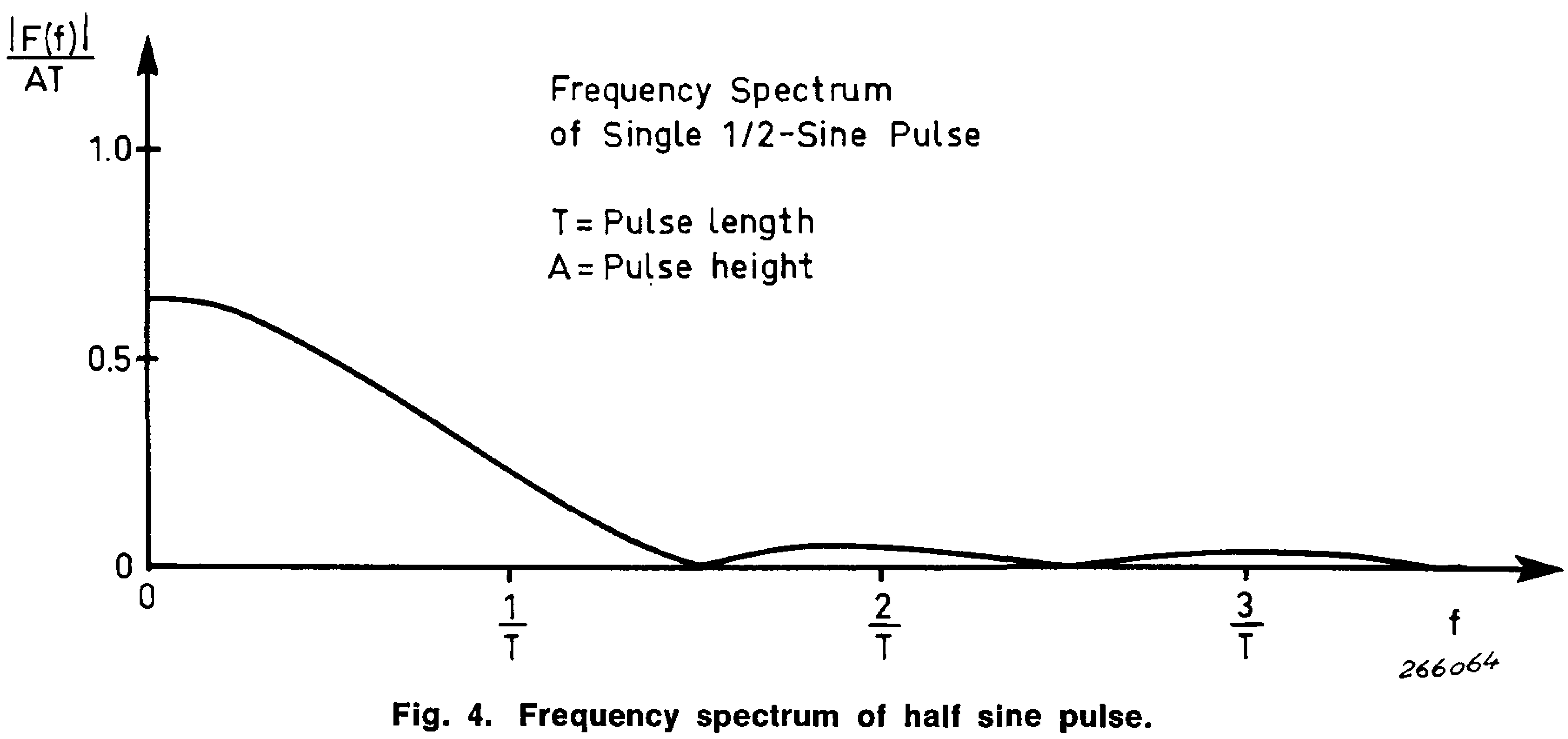


Fig. 4. Frequency spectrum of half sine pulse.

It is seen from these plots that in general a shock pulse contains energy in all frequency bands from zero to infinite frequency. It is also seen that the spectra are continuous, with no discrete frequency components.
In the expressions for F(f) above, all the expressions within the parallel brackets approach unity as f goes to zero, so that at low frequencies the spectrum is equal to the area (amplitude-time integral) of the shock pulse, irrespective of the pulse shape. This fundamental relationship is of considerable practical importance, for example in shock testing. It means that as long as the shock pulse is short compared with the periodic time of the mechanical system on which it acts, the severity of the shock is determined by the area of the shock pulse alone.

Shock Spectra (Maximum Response Spectra)

Another useful concept in the description of shock pulses is the *shock spectrum.* This is obtained by letting the pulse waveform in question be applied to a linear single-degree of freedom system and to find the response of this system as a function of time. If we have a whole series of such single-degree of freedom systems tuned to different resonance frequencies, we can plot for example the absolute maximum response of these as a function of natural frequency. The resulting plot constitutes a shock spectrum.
Various types of shock spectra are used, depending upon application of the information obtained. These may be the *initial* shock spectrum which is obtained from the maximum response while the shock pulse is acting, and the *residual* shock spectrum which is obtained from the maximum response after the pulse has occurred.
Other definitions may be the absolute maximum or *maximax* shock spectrum which is plotted on the basis of the maximum response without regard to time, and the absolute negative maximum shock spectrum which is obtained by considering the maximum response of the single-degree of freedom system in the negative direction. Other definitions may be thought of or invented if required.
The amount of damping in the single-degree of freedom system is usually taken to be zero, but for special applications definite damping factors may be employed. Thus there is an infinite number of possible shock spectra for any shock pulse. For relatively small amounts of damping the shock spectra will not be essentially different from the spectra obtained with no damping, since the response for the first few cycles will be virtually identical. Unless otherwise specified, zero damping will be assumed.
The input quantity used is immaterial, if the same quantity is used for the output quantity. Thus only the waveform is important, and the function used may represent acceleration, velocity or displacement, whatever is convenient. It should be noted that the shock spectrum does not depend upon the size of the spring-mass system but only upon the resonance frequency.

For certain applications it may be required to investigate the response of a resonance system mounted on another resonance system subjected to a shock waveform. If a shock spectrum is calculated or measured for such a configuration it is called a secondary shock spectrum. It depends not only upon the resonance frequencies of the two single-degree of freedom systems, but also upon their relative physical sizes.

Still higher order spectra may be used but are not often seen in practice. In fact primary order spectra are nearly always used, and a shock spectrum is usually taken to mean a primary shock spectrum.

The essential difference between the Fourier spectrum and the shock spectrum is that the Fourier spectrum describes a waveform, whereas the shock spectrum describes the response of a certain physical system to a waveform. Both are very useful concepts and both may be used for such as:

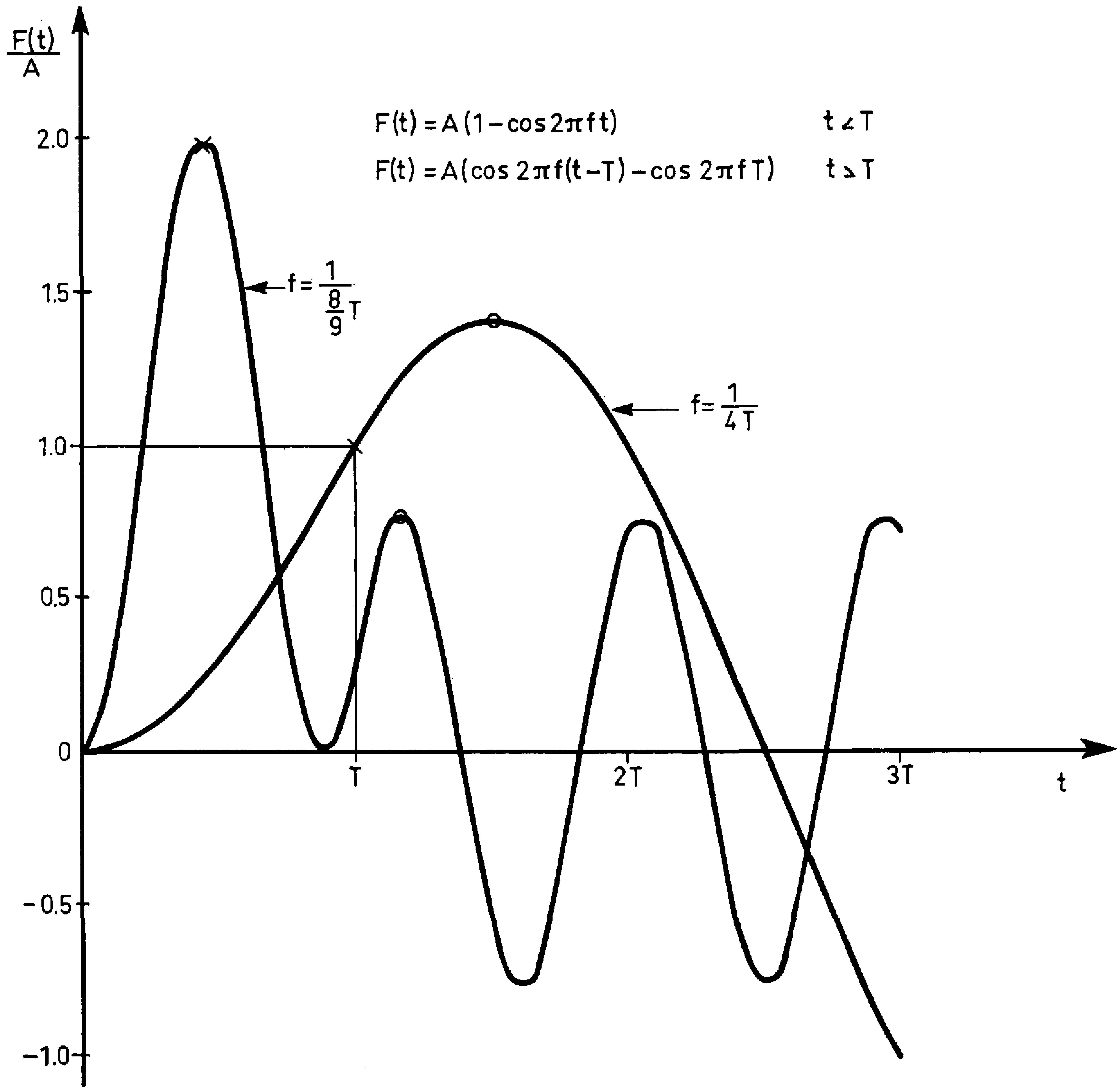


Fig. 5. Time response of simple oscillators to a rectangular shock pulse.

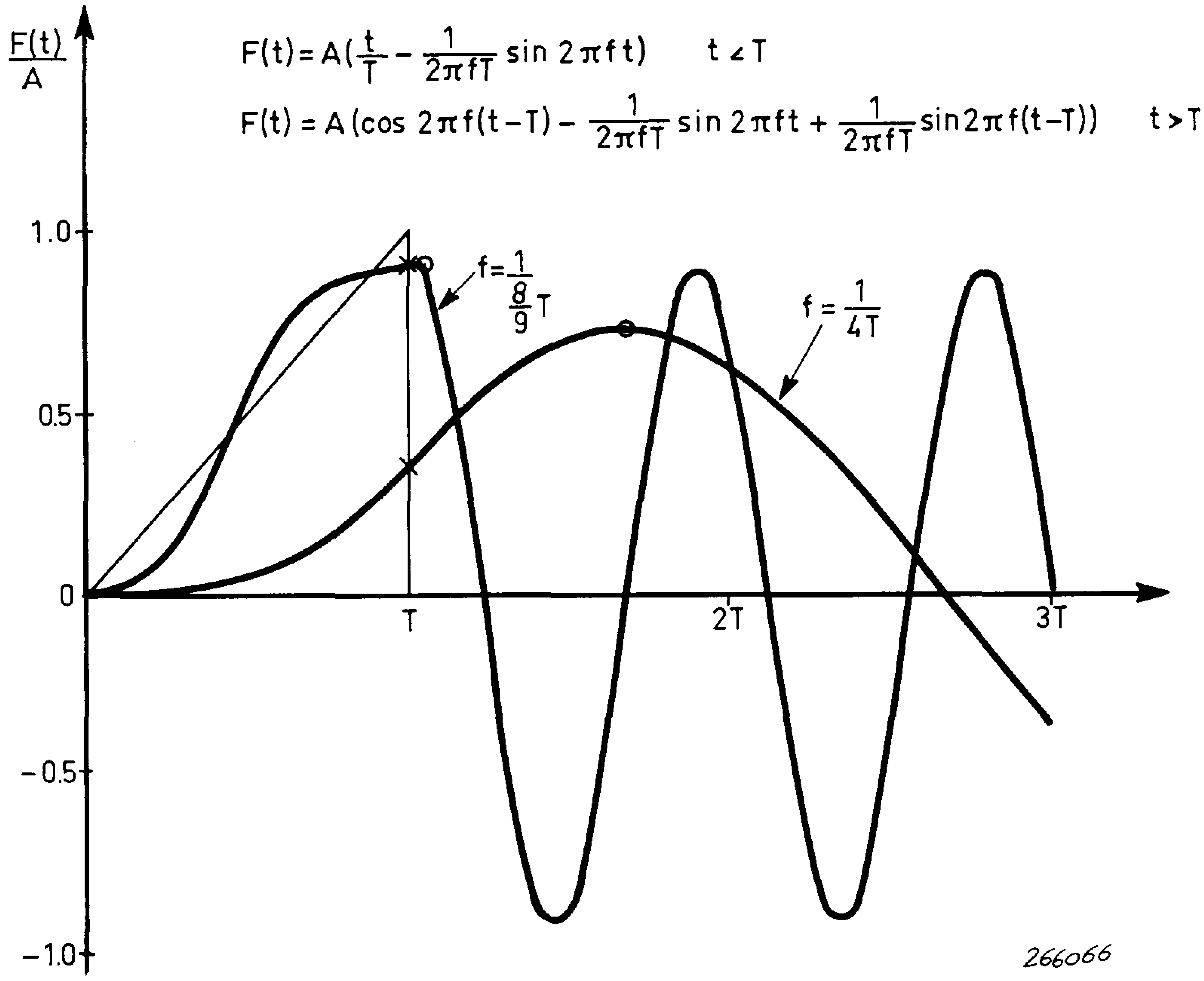


Fig. 6. Time response of simple oscillators to a final peak sawtooth pulse.

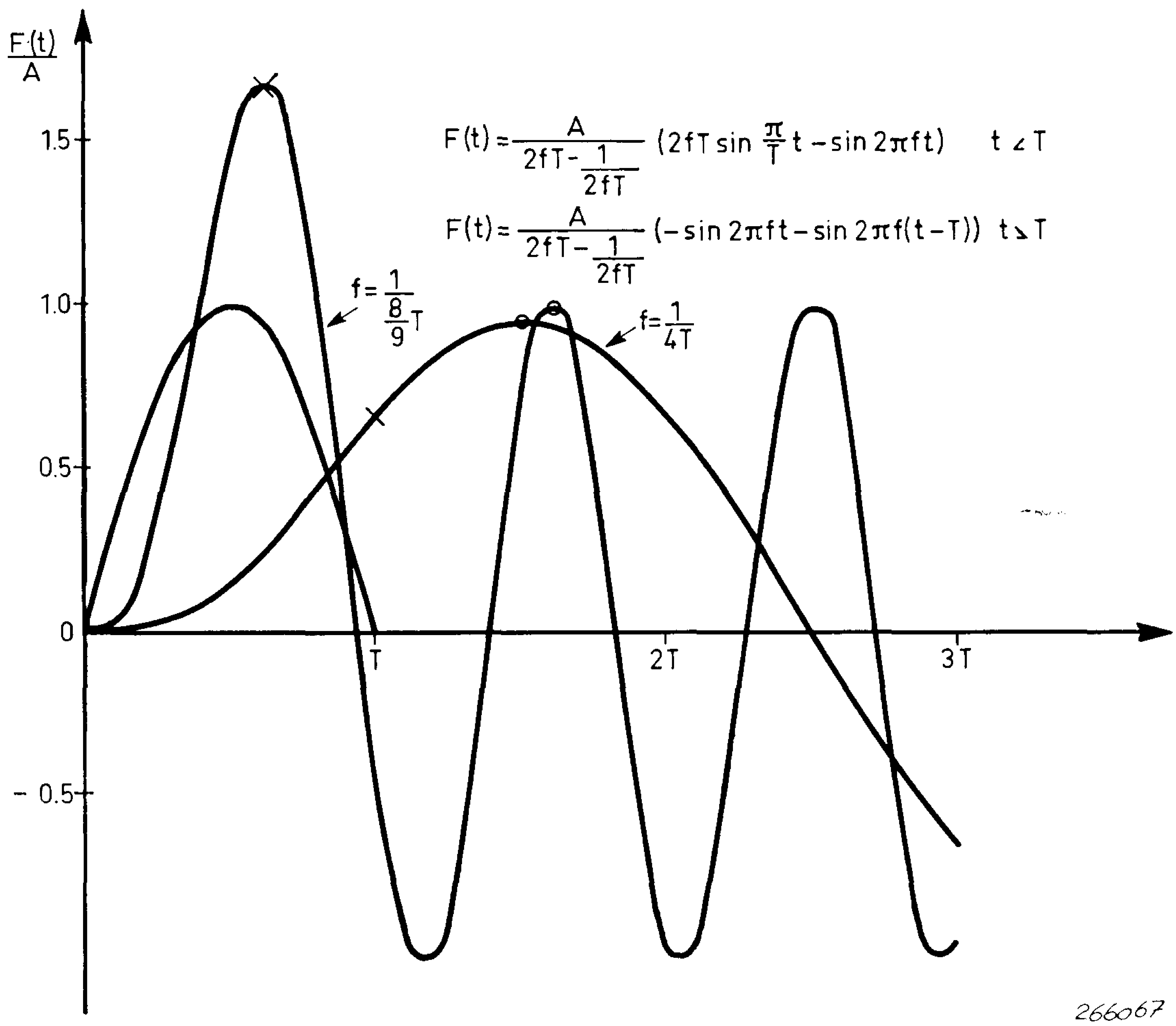


Fig. 7. Time response of simple oscillators to a half sine pulse.

1. Comparing shock severity
2. Selection of shock isolators
3. Shock test specifications
4. A general design tool

The response of a single-degree of freedom system to a shock pulse can be calculated relatively easily for simple waveforms, using for example Laplace transforms.

The time response of two systems of natural frequencies $f = \frac{9}{8\,T}$ and $f = \frac{1}{4\,T}$ to a rectangular, a sawtooth and a half-sine shock pulse of duration T and peak amplitude A is calculated and shown in Fig. 5, 6 and 7. The points marked $\times$ are points on the initial shock spectra and the points marked $\bigcirc$ are points on the residual shock spectra.
Figs. 8, 9 and 10 give the complete shock spectra for these waveforms. The maximas shock spectra are found by taking the highest of the two spectrum values at any frequency.

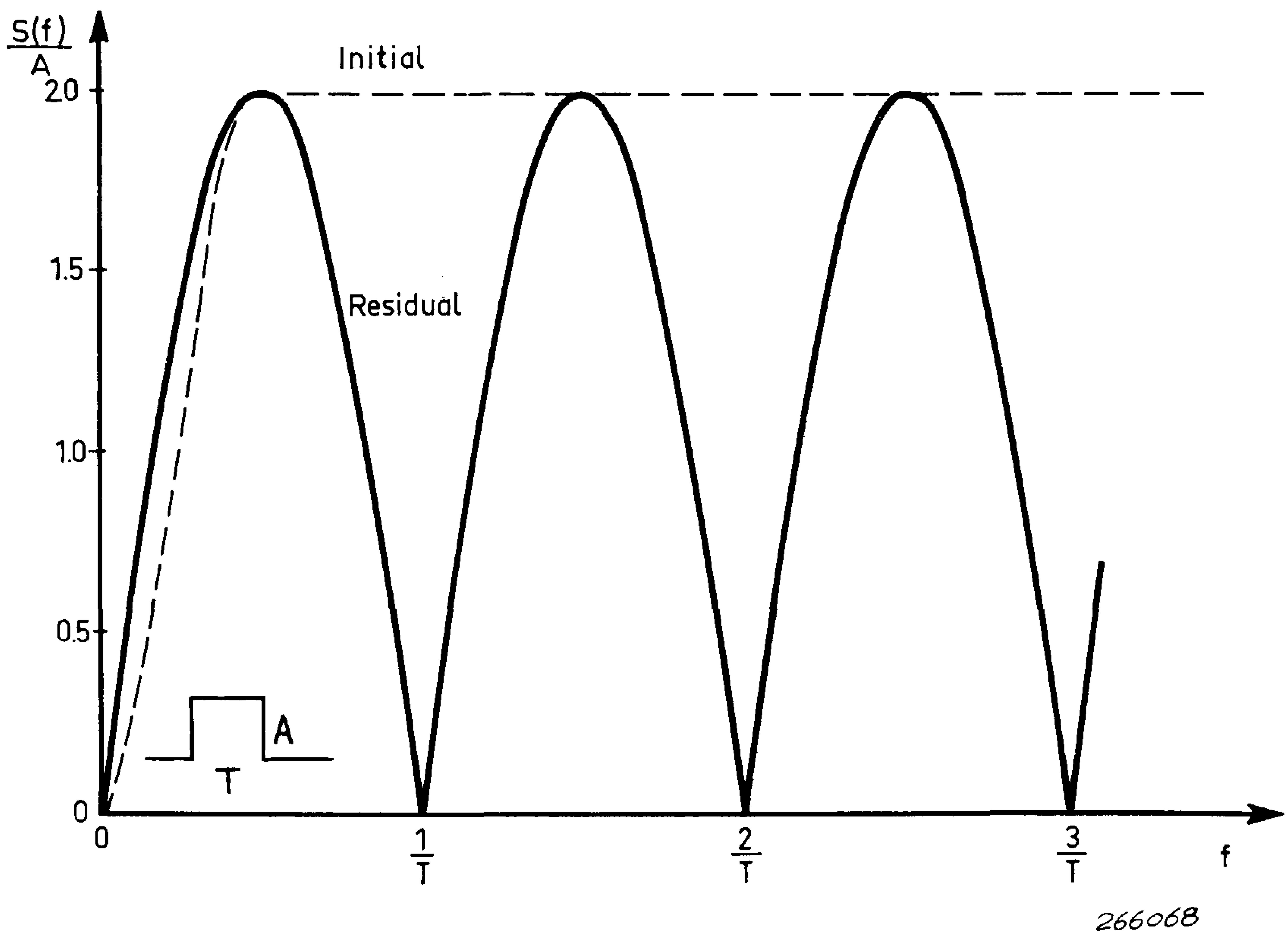


Fig. 8. Shock spectra for rectangular pulse.

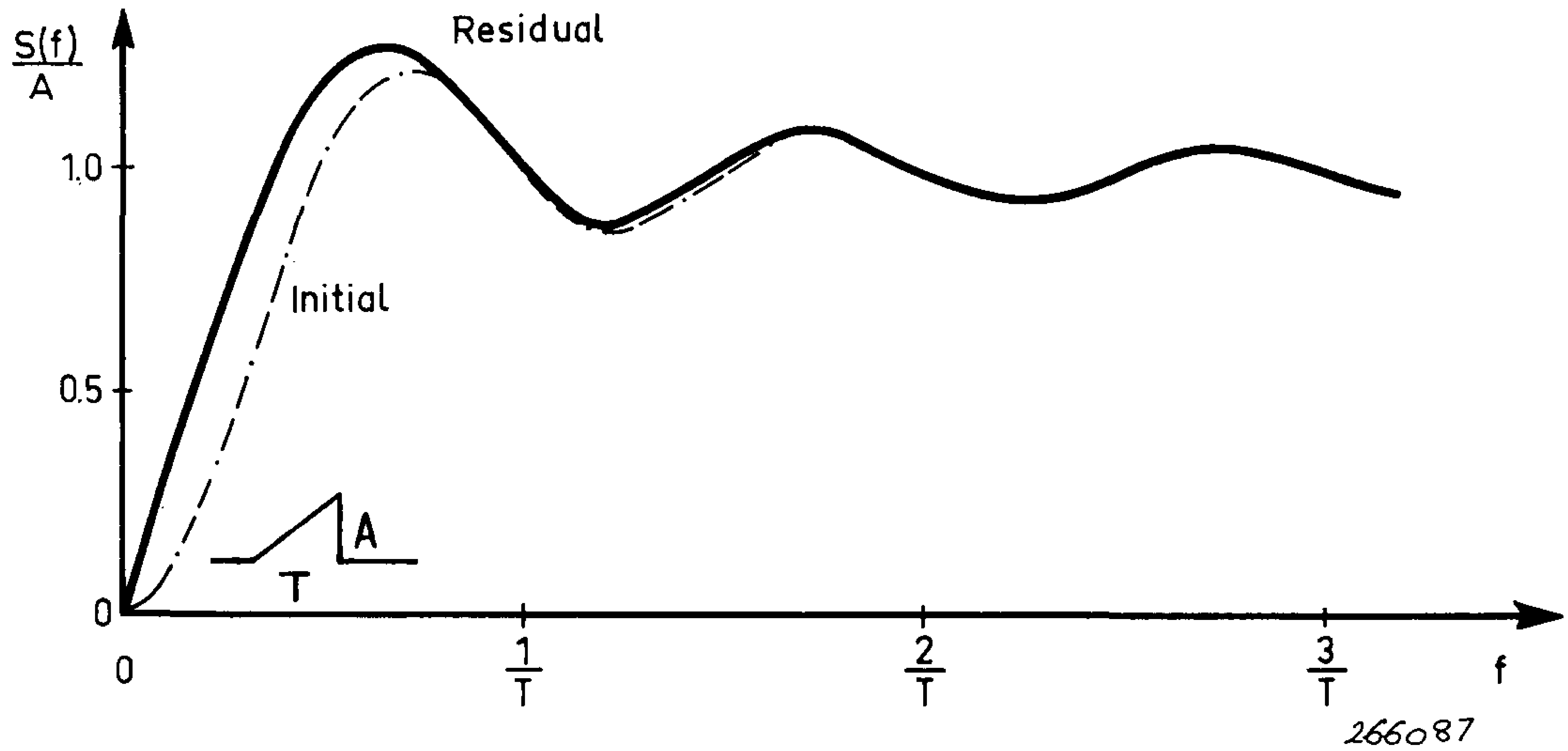


Fig. 9. Shock spectra for final peak sawtooth pulse.

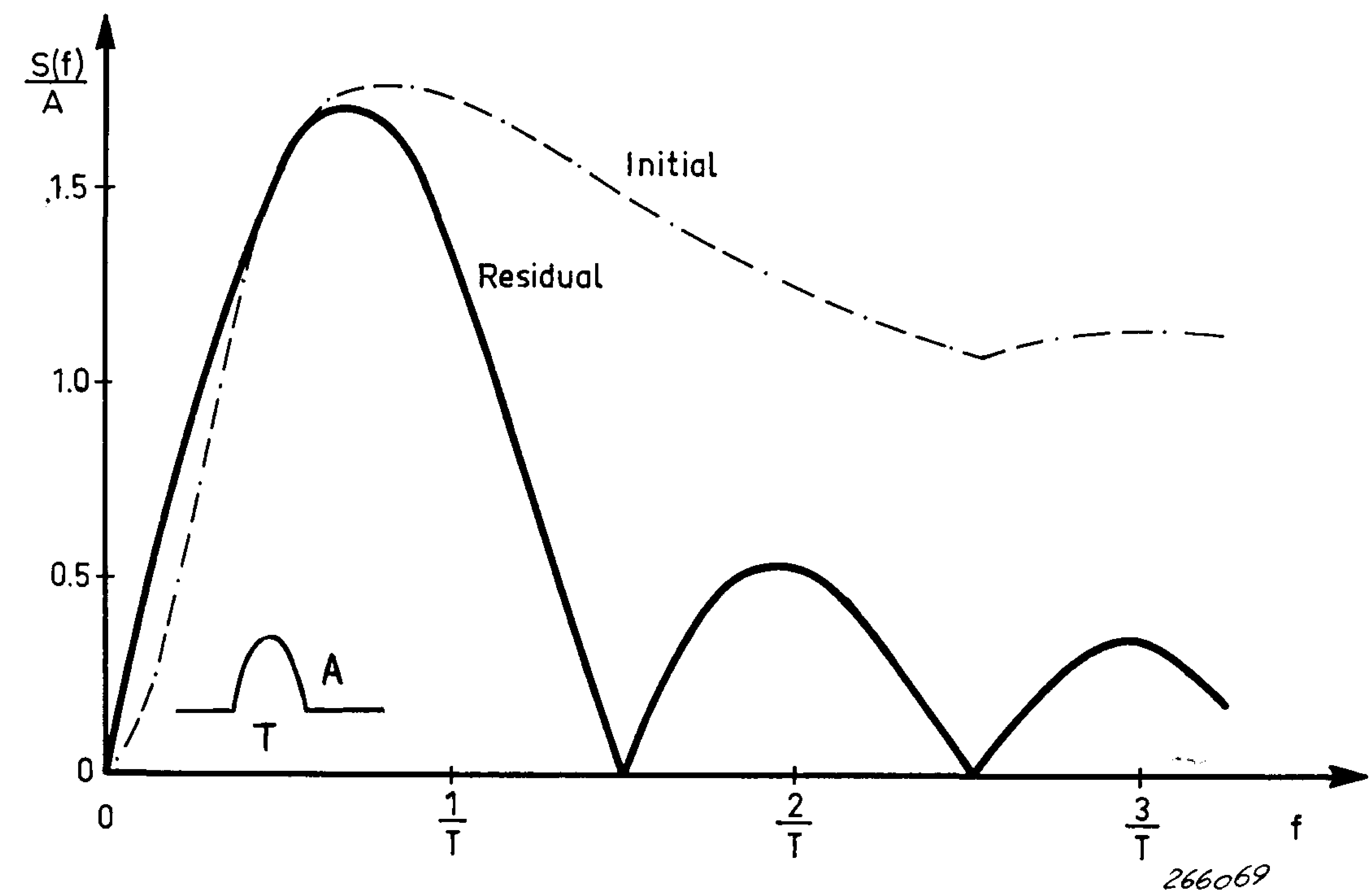


Fig. 10. Shock spectra for half sine pulse.

It is found that the Fourier spectrum and the undamped residual shock spectrum of a pulse are simply related by the formula

$$S(f) = 2 \pi f \, /F(f)/$$

where S(f) is the residual shock spectrum and /F(f)/ is the absolute value of the Fourier spectrum. See Appendix. For other types of shock spectra, such as for example the initial shock spectrum, no such simple relationship exists.

Measurements

It is possible to measure shock spectra directly with the so-called *vibrating reed gauges.* A reed gauge consists of a whole series of reeds tuned to different resonance frequencies. The displacement of these reeds when the instrument is subjected to a certain shock wave can be measured directly. See for example "The Measurement of Acceleration Pulses with the Multifrequency Reed Gauge" by H. Shapiro and D. E. Hudson, Journal of Applied Mechanics, September 1953.
The reed gauge gives a displacement shock spectrum. Acceleration spectra may be obtained by fixing small accelerometers to the mass of the resonance systems as shown in Fig. 11. Of course it is not possible with a reed gauge to have zero damping, but if the quality factor (Q-value) of the resonances is kept high, there will be practically no difference for the first few cycles of movement.

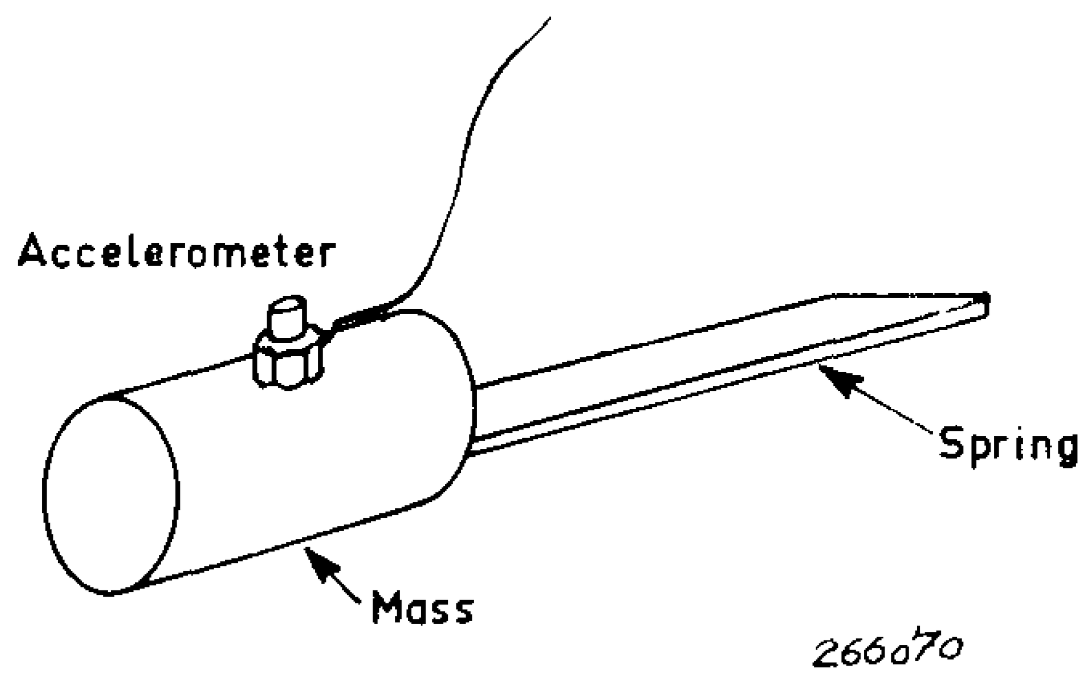


Fig. 11. Small accelerometer fixed to a vibrating reed resonator.

The piezoelectric accelerometer is now used extensively for shock measurements, for several reasons:

1. Most vibration laboratories are equipped with piezoelectric accelerometers, and it is convenient to use the same equipment for both shock and vibration measurements.
2. The accelerometer mass can be made very small, making measurements on lightweight structures possible.
3. Piezoelectric accelerometers have a very wide frequency and dynamic range, which affords great accuracy in the measurements.
4. With modern analog or digital computers the Fourier spectrum or response spectrum of a shock pulse may be obtained relatively quickly, once the waveform is accurately determined.

Measuring System Requirements

When shock pulses are measured, particular requirements have to be fulfilled in order that the output from the transducer and associated circuitry truly follows the amplitude-time history of the shock.

The Fourier spectra given in Figs. 2, 3 and 4 show that in order to give complete information about a shock pulse, the measuring system must have a frequency range from zero (DC) to infinity. *There will always be an error associated with a shock measurement when the measuring system does not have an infinitely wide frequency range.* This error is purely systematic and comes in addition to other errors such as those resulting from calibration inaccuracy, system non-linearity etc.
The required frequency range for a certain percentage error can be calculated for simple pulse shapes. It has been found that the rectangular pulse shape presents the most severe requirements to the measuring system. Any other pulse of the same length will be more accurately measured than the rectangular wave.

Low Frequency Requirements

It has been found that the piezoelectric accelerometer and the first stage of the subsequent electronic instrumentation (usually a preamplifier of some sort) can be represented by a current generator feeding a parallel combination of a resistor and a capacitor as shown in Fig. 12. The resistance is approximately equal to the preamplifier input resistance, and the capacitance is the combined shunt capacitance in the circuit, i.e. accelerometer, cable and preamplifier input capacitance in parallel. The current generated is dQ/dt, where Q is the charge induced on the piezoelectric surfaces.

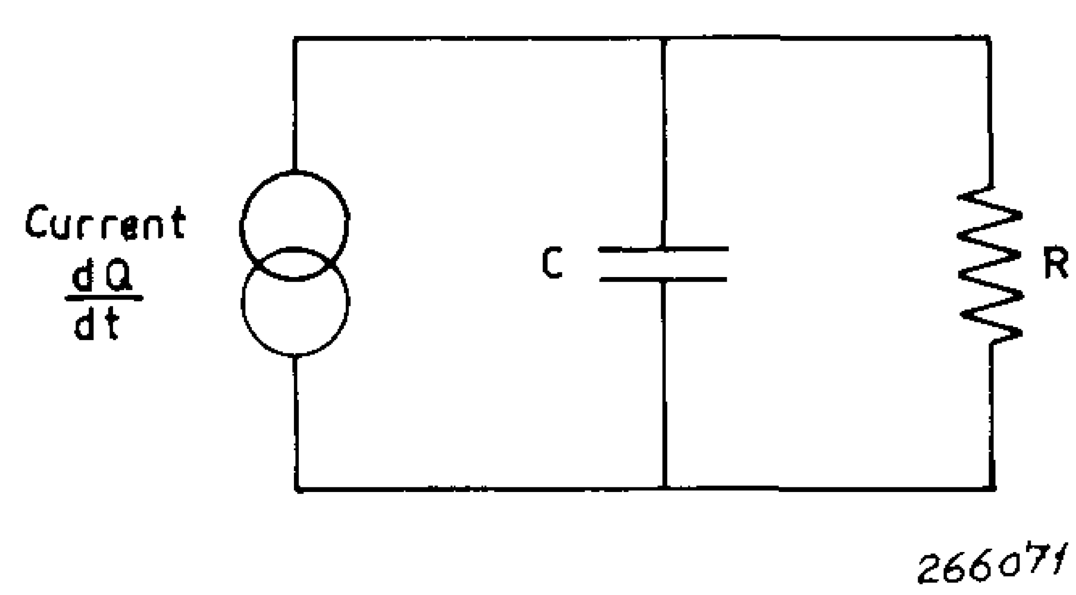


Fig. 12. Equivalent circuit of piezoelectric accelerometer and preamplifier input stage.

The voltage across this system, which is the input voltage to the preamplifier, can be calculated for simple pulse shapes for example by using Laplace transforms. Fig. 13 gives the general output waveshape for a rectangular input pulse. Due to insufficient low-frequency response (no DC response) the peak value cannot be held and at the end of the pulse the voltage has dropped an amount $1 - e^{-T/RC}$ where T is the pulse duration. The undershoot just after the pulse has occurred is also equal to this value.
If similar calculations are made for the half sine and the terminal peak sawtooth pulse, their general shapes will be as shown in Figs. 14 and 15. It is found that the reduction of the peak and the undershoot after the pulse has occurred are less pronounced with these waveforms.
In order to reduce the error to 5 % for example, the factor $e^{-T/RC}$ should be 0.95 or larger for the rectangular pulse. This gives a value for T/RC of about

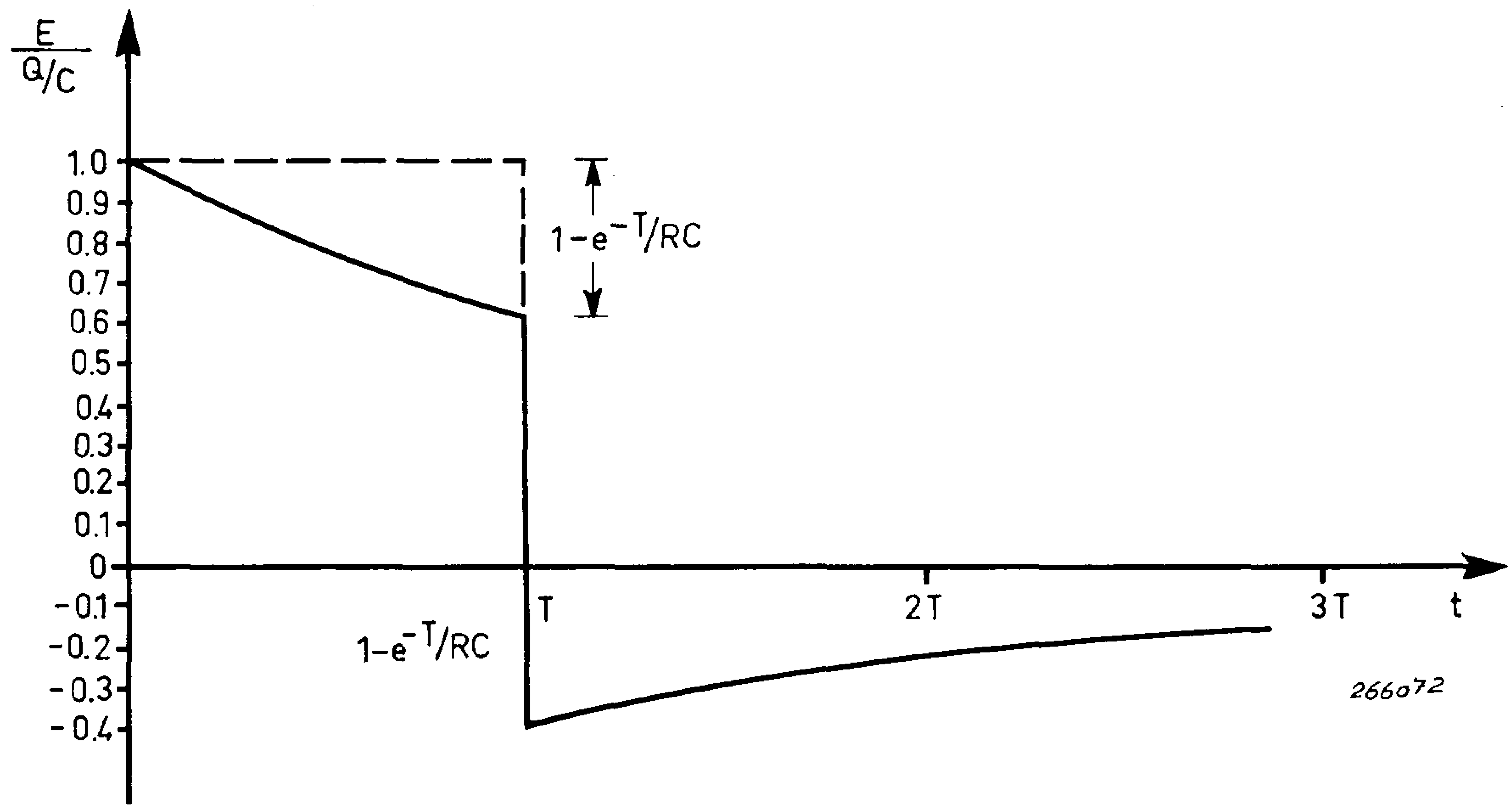


Fig. 13. Preamplifier output voltage versus time for a rectangular acceleration input to accelerometer.

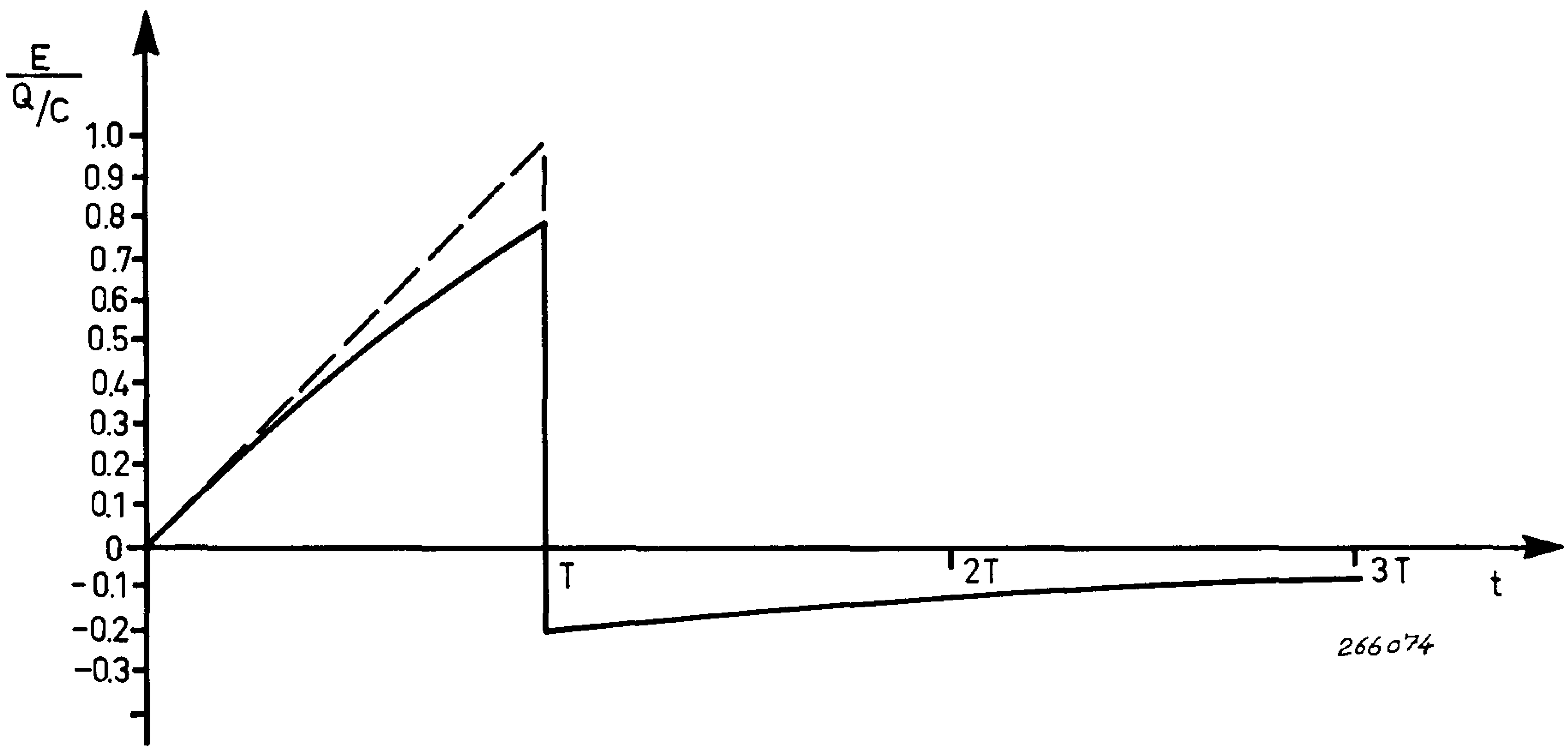


Fig. 14. Preamplifier output voltage for terminal peak sawtooth pulse.

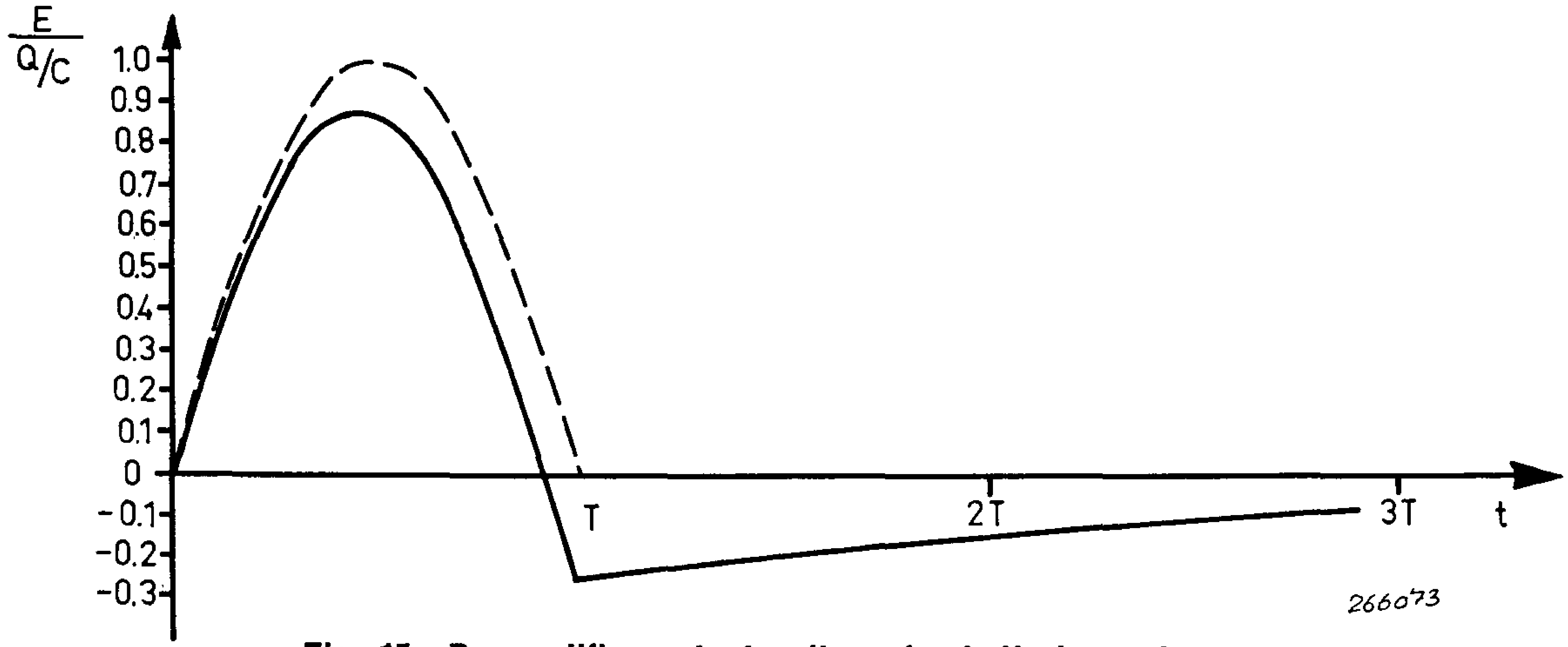


Fig. 15. Preamplifier output voltage for half sine pulse.

o.05, i.e. the time constant, RC, of the measuring system should be twenty times the duration of the pulse. For the half-sine and final peak sawtooth

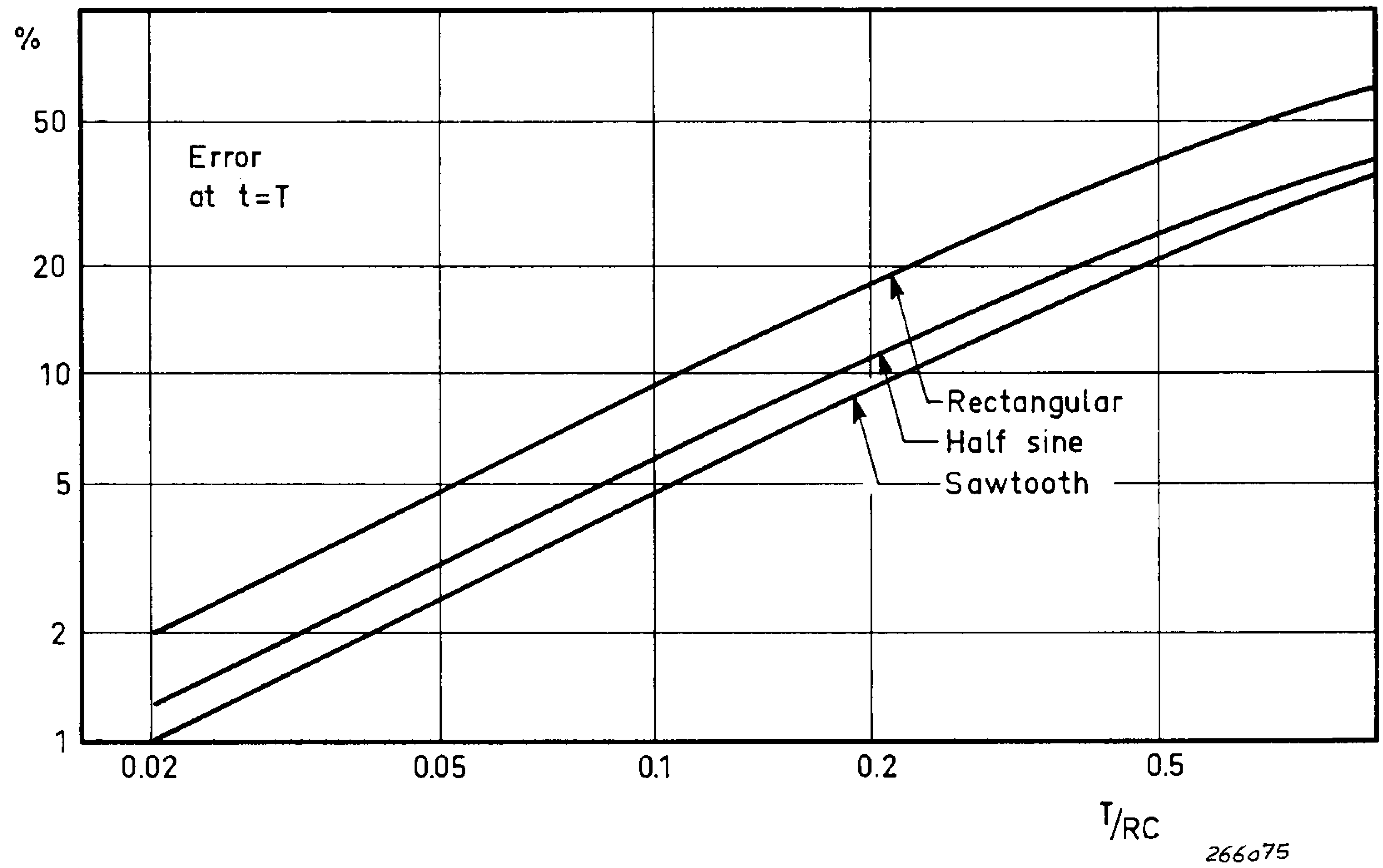


Fig. 16. Error resulting from insufficient low frequency response for rectangular, final peak sawtooth and half sine pulse, referred to peak value of the input.

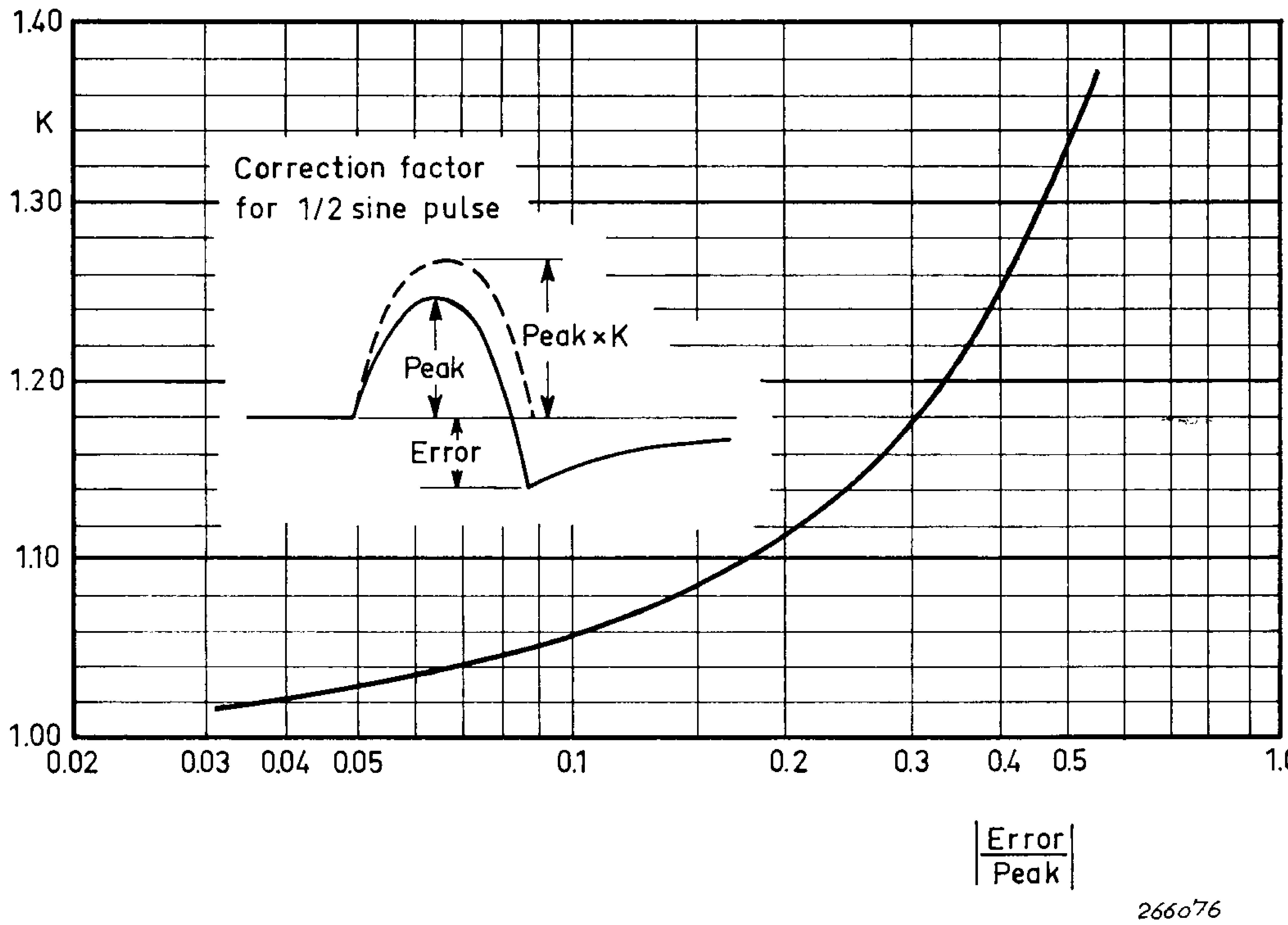


Fig. 17. Correction to peak value for half sine pulse based on the ratio undershoot/ apparent peak.

pulses the RC time constant should be about twelve and nine times the duration of the pulse respectively, for the same error.
Fig. 16 shows the value of the undershoot for these three pulse shapes as a function of T/RC. From this can be deduced that the error for any waveform is always smaller than the ratio T/RC.
If the shock pulse does not have its maximum value at the beginning, i.e. for a half sine or a final peak sawtooth pulse, there will also be a reduction of the peak value of the output. For the final peak sawtooth pulse the reduction in peak value is equal to the undershoot, whereas for a sinusoidal pulse it is approximately equal to half the undershoot, for reasonably small undershoots. A complete correction curve is shown in Fig. 17.

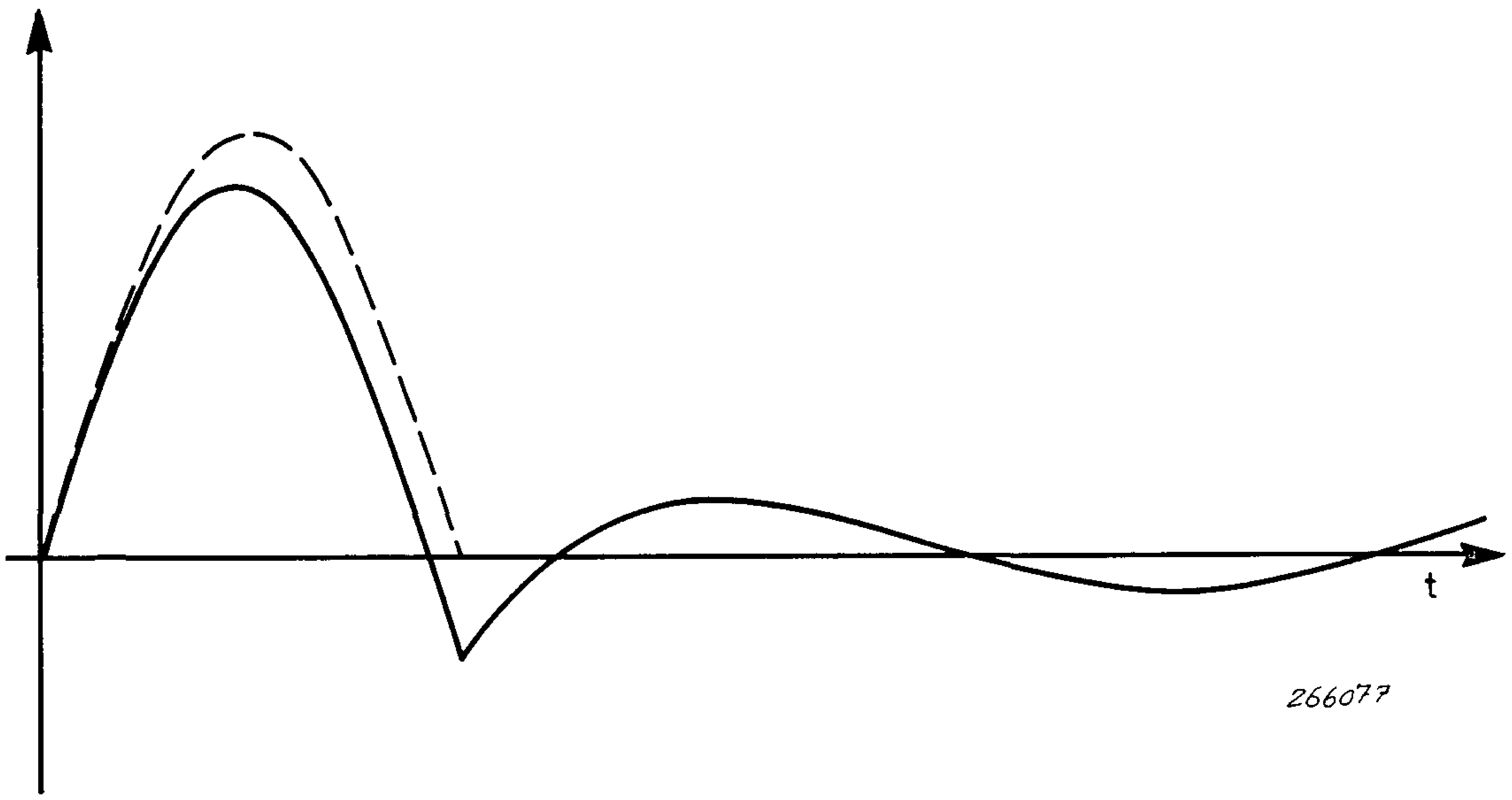


Fig. 18. Possible output waveshape for half sine input when the low frequency cut-off is not the simple 6 dB/octave.

Whenever the low frequency cut-off of the measuring system is determined by some factor other than the RC constant of the input circuit, for example by an amplifier cut-off, the analysis is not quite as simple as that indicated above. For a six dB per octave cut-off the error will be about the same as before if the factor RC is substituted by the expression $\frac{1}{2 \pi f_c}$ where f_c is the -3 dB cut-off frequency. For sharper cut-off rates, for example 12 dB per octave, an oscillatory motion may appear after the pulse in the output, as shown in Fig. 18. Fundamentally, however, the error is still due to insufficient low frequency response, and can only be reduced by extending the frequency range of the instrumentation downwards.

High Frequency Requirements

Referring again to the Fourier spectrum of a rectangular pulse as shown in Fig. 2, it is seen that most of the energy of the pulse is contained in the frequency band from zero to 1/T. In order to handle such a pulse therefore, the system must have an upper frequency limit of at least 1/T, i.e.

$$f_H \geqq \frac{1}{T} \text{ or } f_H T \geqq 1$$

where f_H is the upper frequency limit and T is the pulse length.

The relation $fT \geqq 1$ expresses a fundamental principle, similar to the "uncertainty principle" in quantum mechanics, and is met with in all areas of transmission theory. Its general form is $\Delta f \, \Delta t \geqq 1$ where Δf is the bandwidth of a signal and Δt is its duration.

Applied to a pulse it expresses the fact that the sharper the pulse is determined in time (i.e. the shorter the pulse) the more diffuse it will be in frequency and vice versa. In order for a pulse to have a single frequency ($\Delta f \rightarrow 0$) it must have an infinite duration. Similarly, an infinitely narrow pulse ($\Delta t \rightarrow 0$) has a flat frequency spectrum from zero to infinity.

Not only the amplitude characteristic but also the phase characteristic of the measuring system must be flat up to at least 1/T in order to avoid distortion of the waveform. The phase characteristic of electronic amplifiers is usually reasonably flat up to about one tenth of the cut-off frequency, so that an upper frequency limit of 10/T is adequate for most applications.

Response of Accelerometers to Shock Waveforms

Piezoelectric accelerometers are usually lightly damped single-degree of freedom systems and their response to transient vibrations depends upon their resonance frequency and damping (Ref. 7).

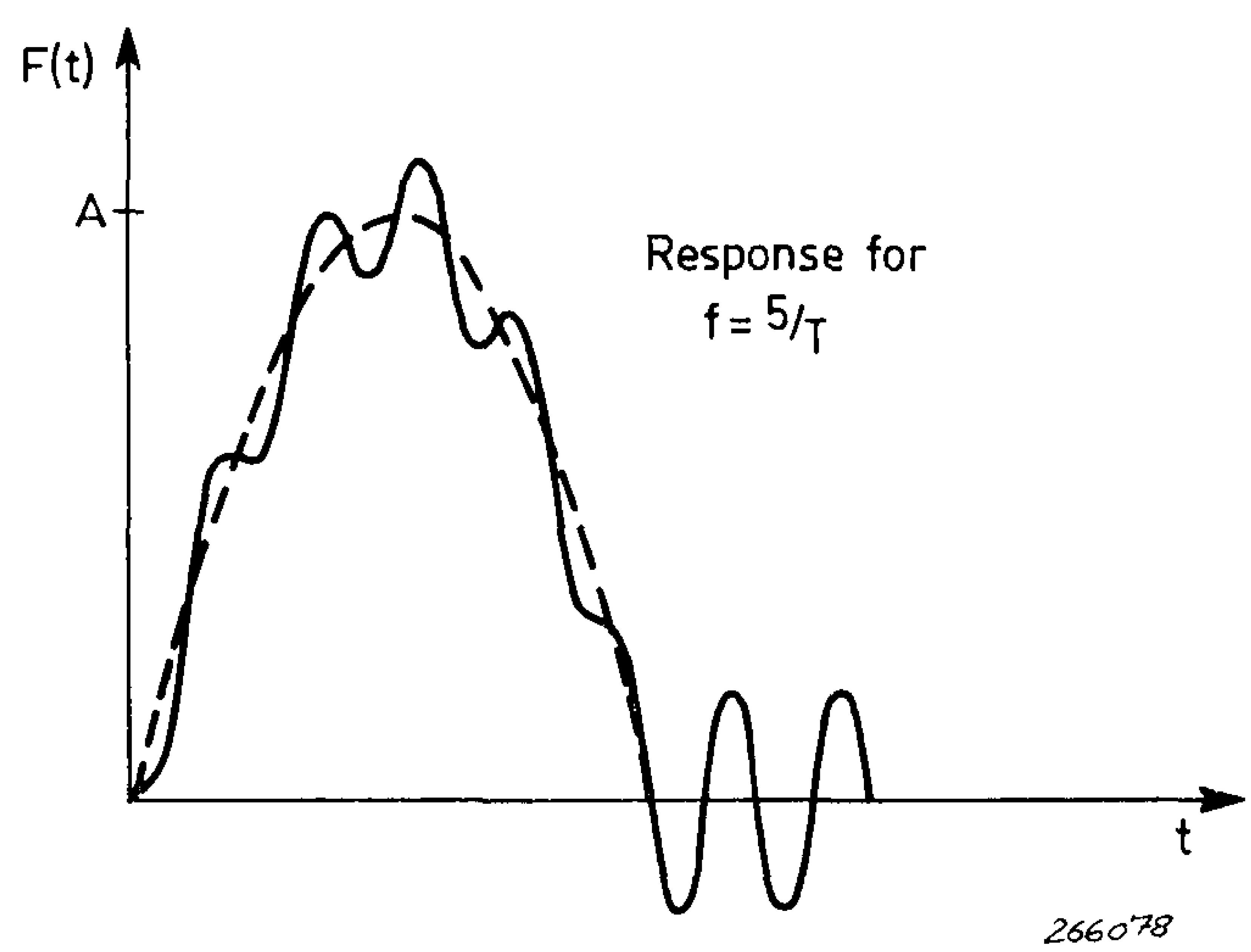


Fig. 19. Response of an undamped accelerometer to a half sine input of duration five times the natural period of the accelerometer.

The response of an undamped accelerometer to a half sine pulse acceleration of duration five times the natural period of the accelerometer is shown in Fig. 19. It is seen that the mean value of the response follows the exciting waveform but there is an oscillatory signal superimposed, which is due to the

accelerometer resonance. This high frequency signal can be filtered out by a low pass filter, but it is necessary to take into account the phase characteristic of the filter. No significant phaseshift must be introduced within the important frequency band of the pulse.
In practice the phase shift of a low pass filter is usually negligible below one tenth of its 3 dB cut-off frequency. As seen before, the main information about a pulse is contained within the frequency range 0 to 1/T where T is the duration of the pulse, so that the filter cut-off frequency should be greater than 10/T. This of course requires that the accelerometer resonance frequency is well above 10/T, depending upon the sharpness of the filter cut-off. See Fig. 20.

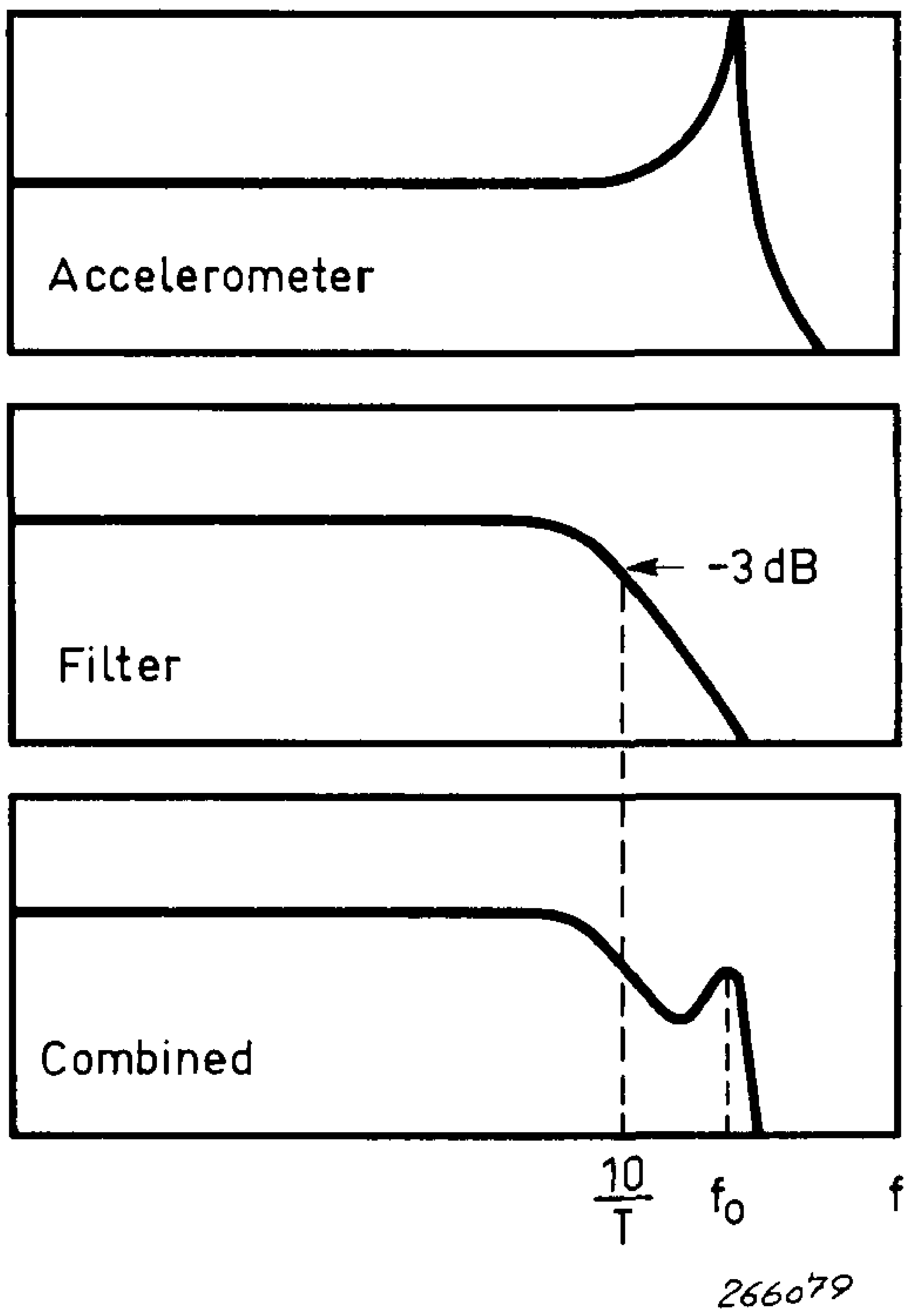


Fig. 20. Use of low-pass filter to eliminate the effect of accelerometer resonance.

Frequency Response of Total System

From the foregoing discussion the following frequency response is required from a shock measuring system for a general pulse waveform:
Low frequency requirements:

$$RC \geqq 20\,T$$

$$f_L \leqq \frac{0.008}{T}$$

High frequency requirements:

$$f_H > \frac{10}{T}$$

These limits are calculated for a rectangular waveform with a maximum of 5 % low-frequency error and a tolerable rounding of corners due to high fre-

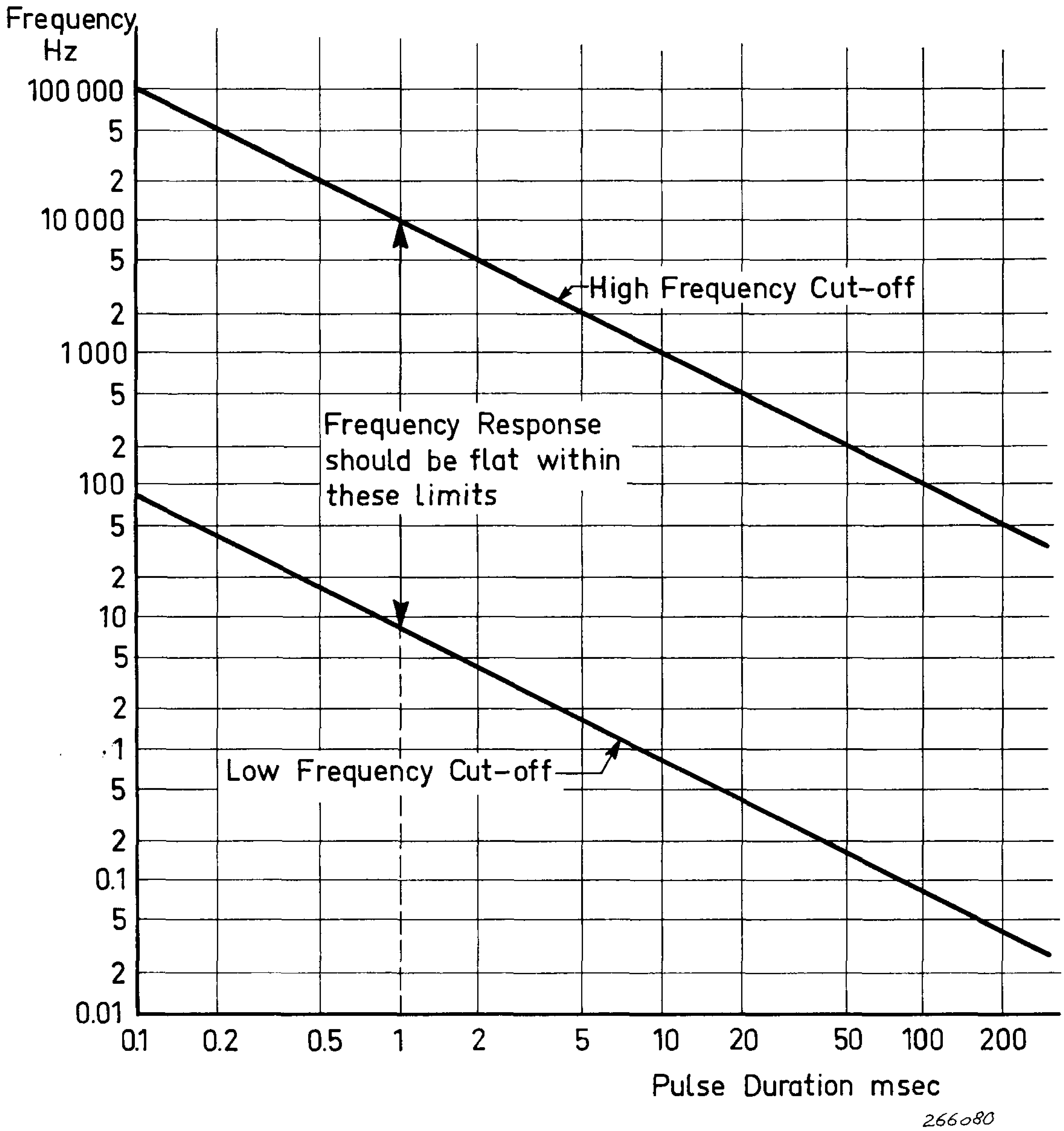
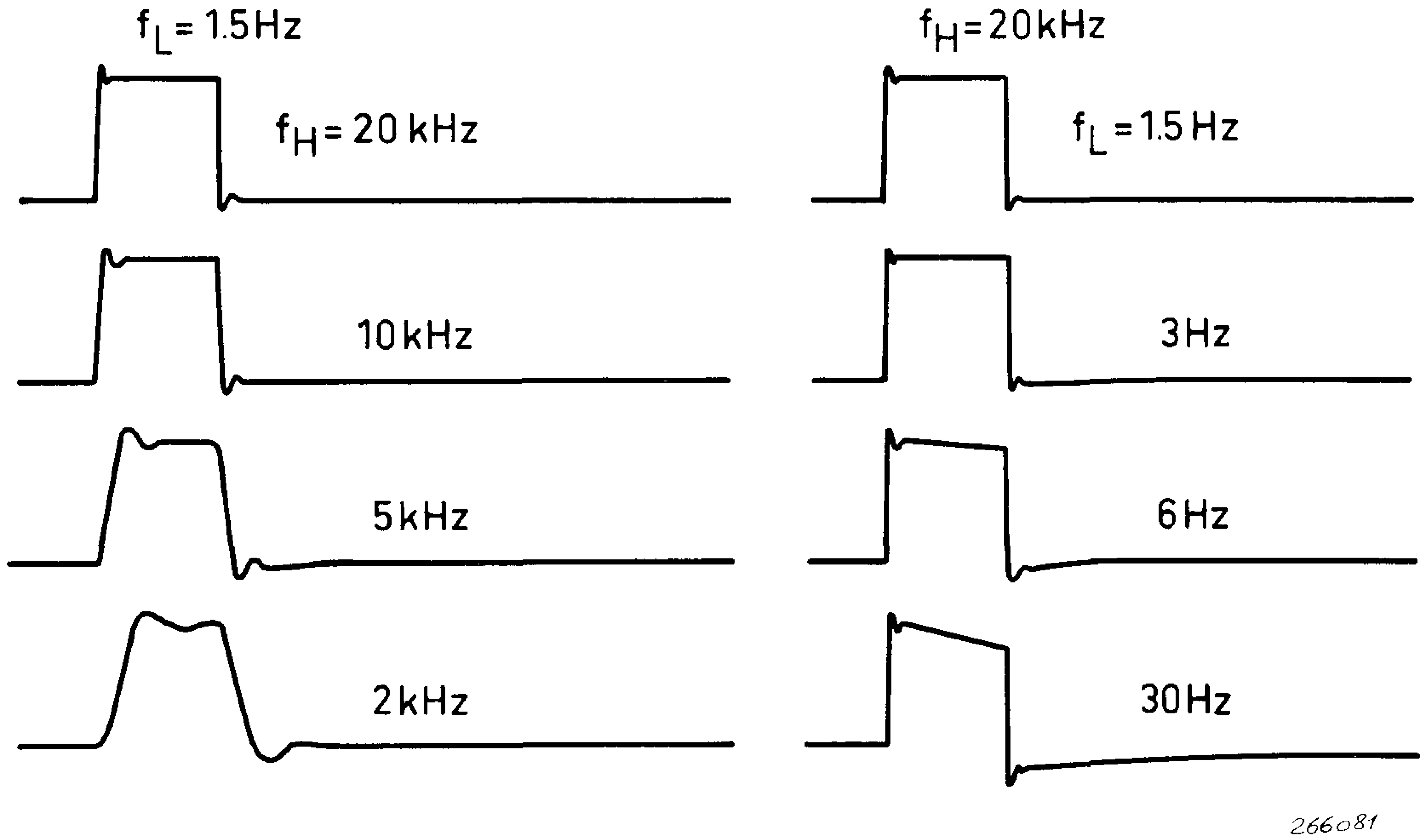


Fig. 21. Chart for finding required instrumentation frequency range when pulse length is known.

quency limitation. For practical waveforms a narrower frequency range may be acceptable. In particular the low frequency limit can be made considerably higher, if one is aware of the reason for the undershoot at the end of the pulse. The high frequency limit is generally not difficult to achieve with modern instrumentation.

Fig. 21 shows graphically the necessary frequency range for variable pulse length, (rectangular waveform) and Fig. 22 shows the results obtained in practice when measuring such a waveform using variable high and low frequency limits.

Pulse Lenght 1 msec



Conclusion

Complete information about mechanical shocks can only be obtained with a measuring system with a wide frequency range. The modern piezoelectric accelerometer with electronic instrumentation is very well suited for recording shock waveforms, and subsequent analysis may be carried out with analog or digital techniques.

The frequency range of modern accelerometers and measuring electronics can be made wide enough to reduce to insignificance the errors associated with inadequate frequency response.

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Appendix

The Fourier Spectrum

The spectrum of a transient wave is expressed by the Fourier integral. This is derived from the ordinary Fourier representation of periodic waves in the following way:

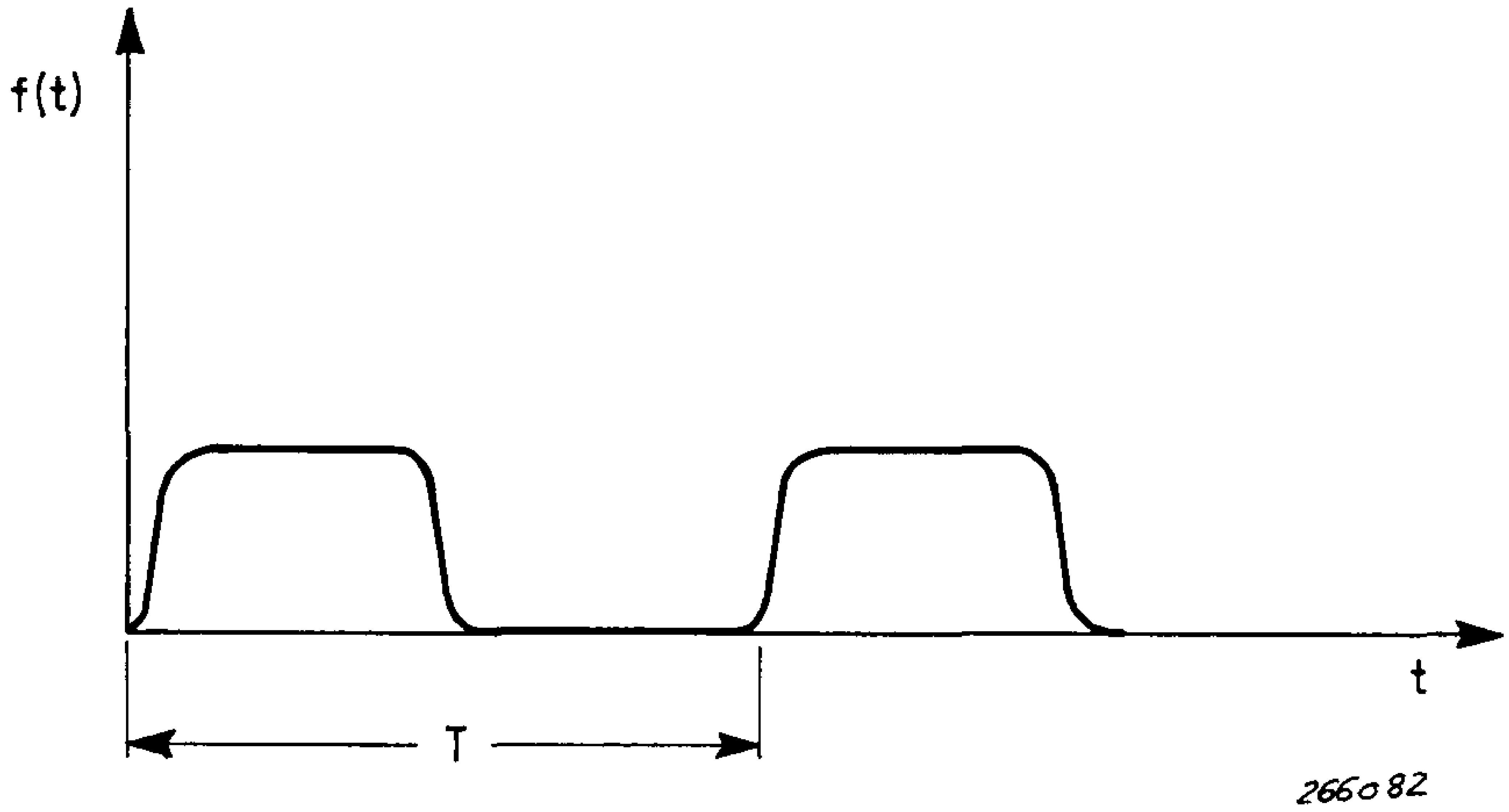

Fig. A1.

Assume a periodic wave f(t) of period T as shown in Fig. A1. This can be represented by the Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{i2n\pi f_0 t}$$

where $$C_n = f_0 \int_{-T/2}^{T/2} f(t)\, e^{-in2\pi f_0 t}\, dt$$

and $$f_0 = \frac{1}{T}$$

C_n are called the frequency components of the wave and are spaced at equal intervals 1/T on the frequency scale.

The whole expression for f(t) is

$$f(t) = \sum_{n=-\infty}^{\infty} f_0\, e^{in2\pi f_0 t} \int_{-T/2}^{T/2} f(t)\, e^{-in2\pi f_0 t}\, dt$$

Let us now increase the period of the wave such that $T \to \infty$. This makes f_o vanishingly small. The components C_n are then spaced infinitesimally close to each other, so that a continuous function is generated.

Let us denote nf_o by a variable f and f_o by df.

When these values are substituted into the expression for f(t) the sum turns into an integral and we obtain

$$f(t) = \int_{-\infty}^{\infty} df\, e^{j2\pi ft} \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi ft}\, dt$$

or $$f(t) = \int_{-\infty}^{\infty} F(f)\, e^{j2\pi ft}\, df$$

where $$F(f) = \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi ft}\, dt$$

The continuous function F(f) is called the Fourier spectrum of the function f(t) and the integral on the right is called the Fourier integral.

For a rectangular pulse of amplitude A and duration T the spectrum is

$$F(f) = \int_{-T/2}^{T/2} A\, e^{-j2\pi ft}\, dt$$

$$= \frac{A}{j\, 2\pi f} \left(e^{j\pi fT} - e^{-j\pi fT}\right)$$

$$F(t) = \frac{A\,T}{\pi f T} \sin \pi f T$$

Similar calculation gives the Fourier spectra of the other pulse shapes considered in the main text.

Shock Spectra

The response of a single-degree of freedom system can be calculated relatively easily for simple waveforms, using Laplace transforms. Such a system is shown in Fig. A2.

Letting the displacement of the base be $X_1(t)$ and the displacement of the mass be X(t) we have

$$M \frac{d^2 X}{dt^2} = K (X_1 - X)$$

which, using the Laplace operator gives

$$(Ms^2 + K)\, X(s) = KX_1\, (s)$$

or $$X(s) = \frac{K\, X_1\, (s)}{Ms^2 + K} = \frac{\omega_o^2\, X_1\, (s)}{s^2 + \omega_o^2}$$

where $\omega_o = \sqrt{K/M}$ is 2π times the natural resonance frequency of the single-degree of freedom system.

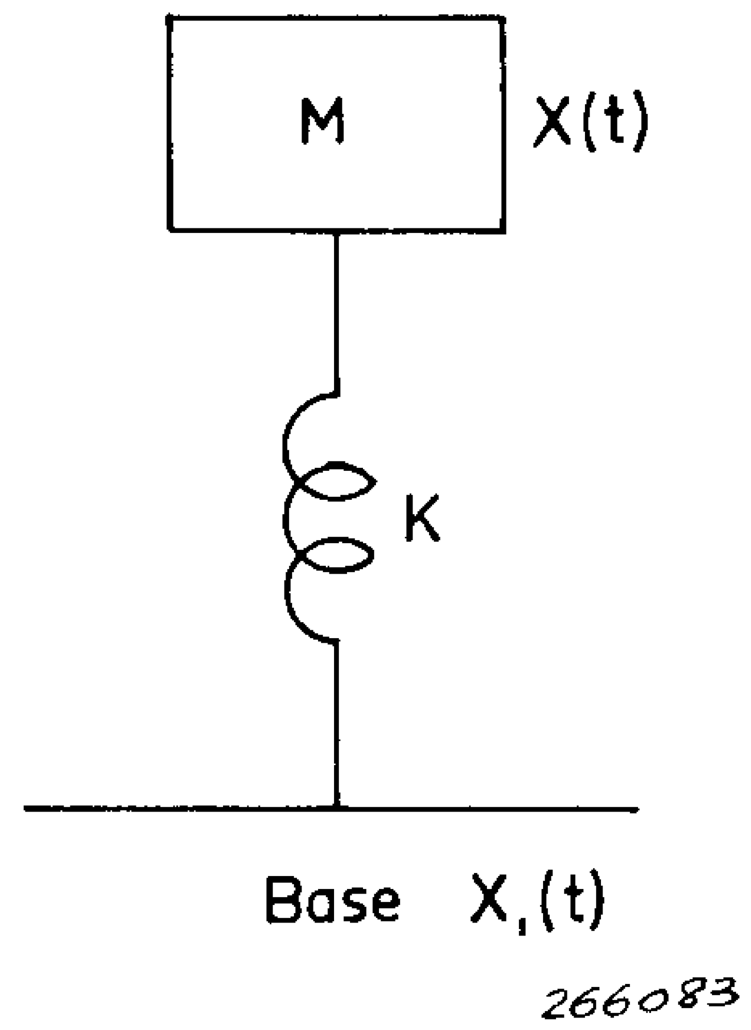

Fig. A2.

Single Rectangular Pulse

The single rectangular pulse is the superposition of two step functions of equal amplitude A and opposite direction, applied with a time delay T, the length of the pulse.

The response of the single-degree of freedom system to such a pulse is

$$X(s) = \frac{\omega_o^2}{s^2 + \omega_o^2}\,\frac{A}{s} = A\left(\frac{1}{s} - \frac{s}{s^2 + \omega_o^2}\right)$$

or $\qquad X(t) = A\,(1 - \cos \omega_o t) \qquad 0 < t < T$

The response for $t > T$ is obtained by subtracting a similar response delayed an amount T, i.e.

$$X(t) = A\,(1 - \cos \omega_o t) - A\,(1 - \cos \omega_o (t - T))$$

$$X(t) = A\,(\cos \omega_o (t - T) - \cos \omega_o t) \qquad t > T$$

The time response for a certain system is then as shown in Fig. A3. The amplitude A^1 is a point on the residual shock spectrum.

At $t = T$, $\quad A\,(1 - \cos \omega_o T) = A^1 \sin \omega_o (T - T^1)$ 1

where T^1 is the imaginary starting point of the free oscillation.

Also the derivative of the response curve is smooth, i.e.

$$A\,\omega \sin \omega_o T = A^1\,\omega \cos\, \omega_o (T - T^1)$$

or $\qquad A \sin \omega_o T = A^1 \cos \omega_o (T - T^1)$ 2

By combination of equations 1 and 2 we obtain

$$A^2\,[(1 - \cos \omega_o T)^2 + \sin^2 \omega_o T] = A^{1\,2}$$

$$A^1 = A\sqrt{2\,(1 - \cos \omega_o T)}$$

$$A^1 = 2\,A \sin \omega_o T/2$$

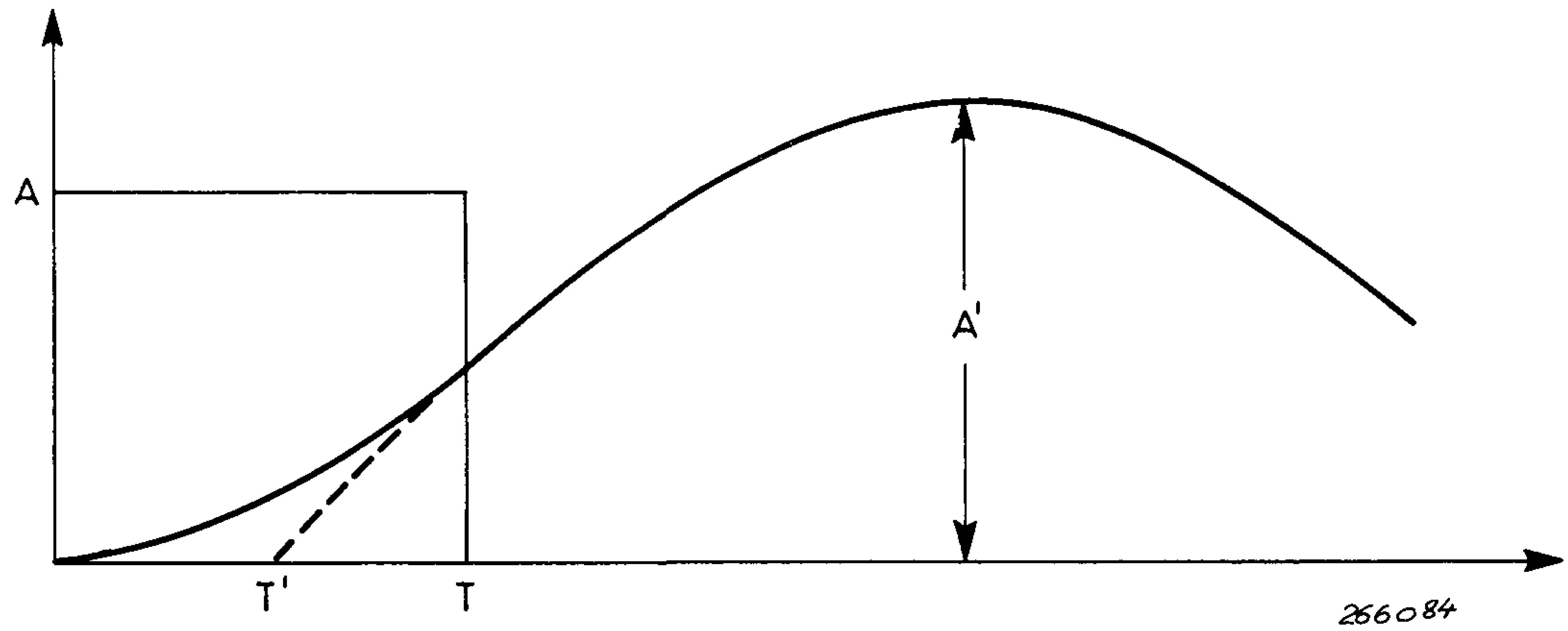

Fig. A3.

Thus the residual shock spectrum for the rectangular pulse is

$$S(f) = 2\,A \sin \omega\, T/2$$

when ω is variable.

The initial shock spectrum is seen intuitively to be

$$S(f) = A\,(1 - \cos \omega\, T) \qquad f \leqq 1/2\,T$$

$$S(f) = 2\,A \qquad f \geqq 1/2\,T$$

Similar procedures give the shock spectra for the other waveforms considered in the main text.

Connection between Fourier Spectrum and Residual Shock Spectrum

It can be demonstrated that there is a definite relationship between the Fourier Spectrum of a shock pulse and its Undamped Residual Shock Spectrum. Thus

$$S(f) = 2\,\pi\, f\, /F(f)/$$

where S(f) is the residual shock spectrum and F(f) is the Fourier spectrum.

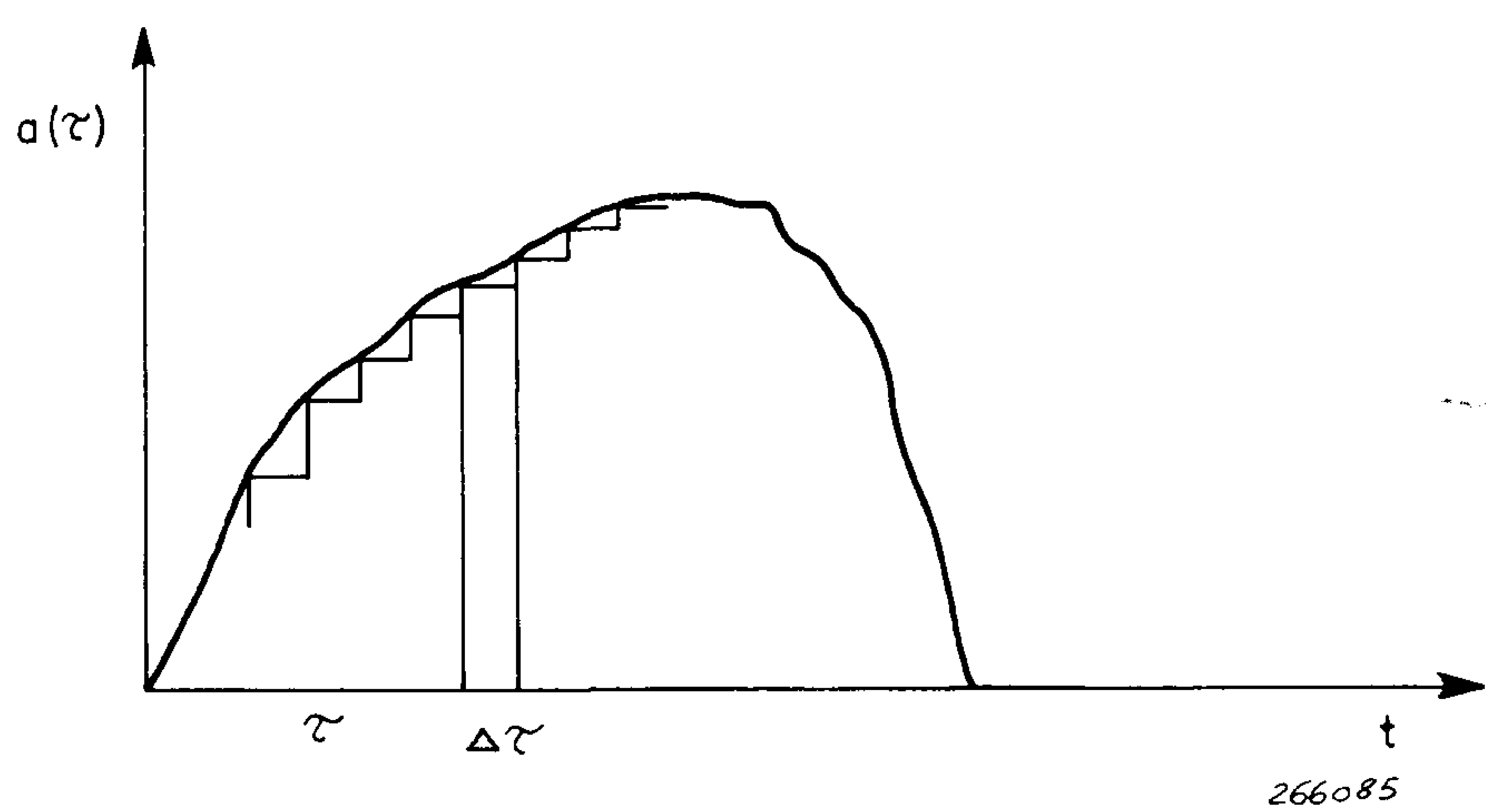

Fig. A4.

Fig. A4 shows an arbitrary acceleration shock amplitude as a function of time. Assuming a linear resonance system, its response to such a shock can be calculated as the superposition of the responses to a number of step functions approximating the shock pulse.

The change in input velocity per step is

$$\Delta v = a(\tau)\,\Delta\tau$$

where $a(\tau)$ is the value of the input acceleration at time τ and $\Delta\tau$ is the width of the step.
The partial velocity response at some time t after the step has occurred is

$$\Delta v_R = h\,(t-\tau)\,a(\tau)\,\Delta\tau$$

where $h\,(t-\tau)$ is the velocity response to a unit velocity step. The total response at a time t after the shock has occurred is then

$$v_R = \Sigma\, h\,(t-\tau)\,a(\tau)\,\Delta\tau$$

Letting the width of the steps $\Delta\tau$ approach zero, the sum turns into an integral

$$v_R = \int_0^T h\,(t-\tau)\,a(\tau)\,d\tau$$

Since the input occurs only during the time 0 to T nothing is changed by changing the limits of integration to $-\infty$ to ∞, i.e.

$$v_R = \int_{-\infty}^{\infty} h\,(t-\tau)\,a(\tau)\,d\tau$$

The velocity response of an undamped resonator to a unit velocity step is the same as the acceleration response to a unit acceleration step. Thus

$$h\,(t-\tau) = 1-\cos 2\,\pi\,f\,(t-\tau)$$

as may be seen from Fig. 5 on page 8.
Thus we have

$$v_R = \int_{-\infty}^{\infty} (1-\cos 2\,\pi\,f\,(t-\tau)\,a(\tau))\,d\tau$$

$$= \int_{-\infty}^{\infty} a(\tau)\,d\tau - \int_{-\infty}^{\infty} a(\tau)\cos 2\,\pi\,f\,(t-\tau)\,d\tau$$

The first integral must be zero for a transient waveform so that

$$v_R = -\int_{-\infty}^{\infty} a(\tau)\cos 2\,\pi\,f\,(t-\tau)\,d\tau$$

This is also an expression for the Fourier spectrum of the shock pulse, except for the minus sign. Thus

$$v_R = /F(f)/$$

and the maximum residual response acceleration is

$$a_R = 2\,\pi\,f\,/F(f)/$$

which is what we set out to demonstrate.

Accelerometers

A cross-section of a modern accelerometer is shown in Fig. A5. It consists of a piezoelectric element on which rests a heavy mass. The mass is preloaded by a stiff spring, and the whole assembly is mounted in a metal housing with a thick base. When the accelerometer is subjected to vibration, the mass will exert a variable force on the piezoelectric element. This force is exactly pro-

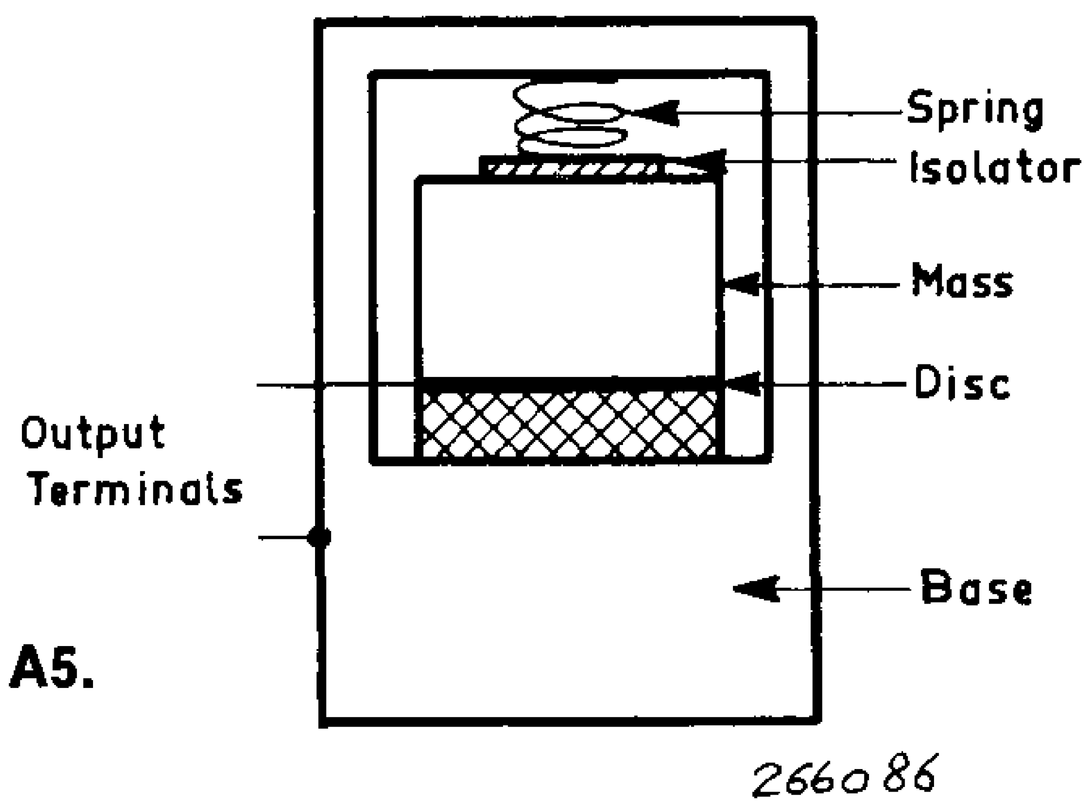

Fig. A5.

portional to the acceleration of the mass. Due to the piezoelectric effect, a variable charge will be developed across the piezoelectric element. This charge is proportional to the force and therefore to the acceleration of the mass. For frequencies much lower than the resonance frequency of the mass on the combined stiffness of the whole accelerometer system, the acceleration of the mass will be virtually the same as the acceleration of the whole transducer, and the charge produced will be proportional to the acceleration to which the transducer is subjected. This charge can be measured electronically at the output terminals and used for accurate determination of vibration amplitude, waveform and frequency.

Relevant specifications for shock measurement for two different Brüel & Kjær accelerometers are:

Accelerometer Type	*4333*	*4336*
Sensitivity, mV/g	14–20	4–6
Mounted resonance, kHz	45	125
Capacity with cable, pF	1000	300
Resistance at 20°C, Mohm	> 20000	> 20000
Max. shock, g	10000	14000
Frequency range (1 dB), Hz	0.5–15000	0.5–40000
Type of connection	Side	Side
Weight, grams	13	2

Brief Communications

Recording of RPM-Changes due to Loading of AC Generators.

Communicated from Ing. Elmar Dorner of Jenbacker Motoren-Werke, Tirol, Austria.

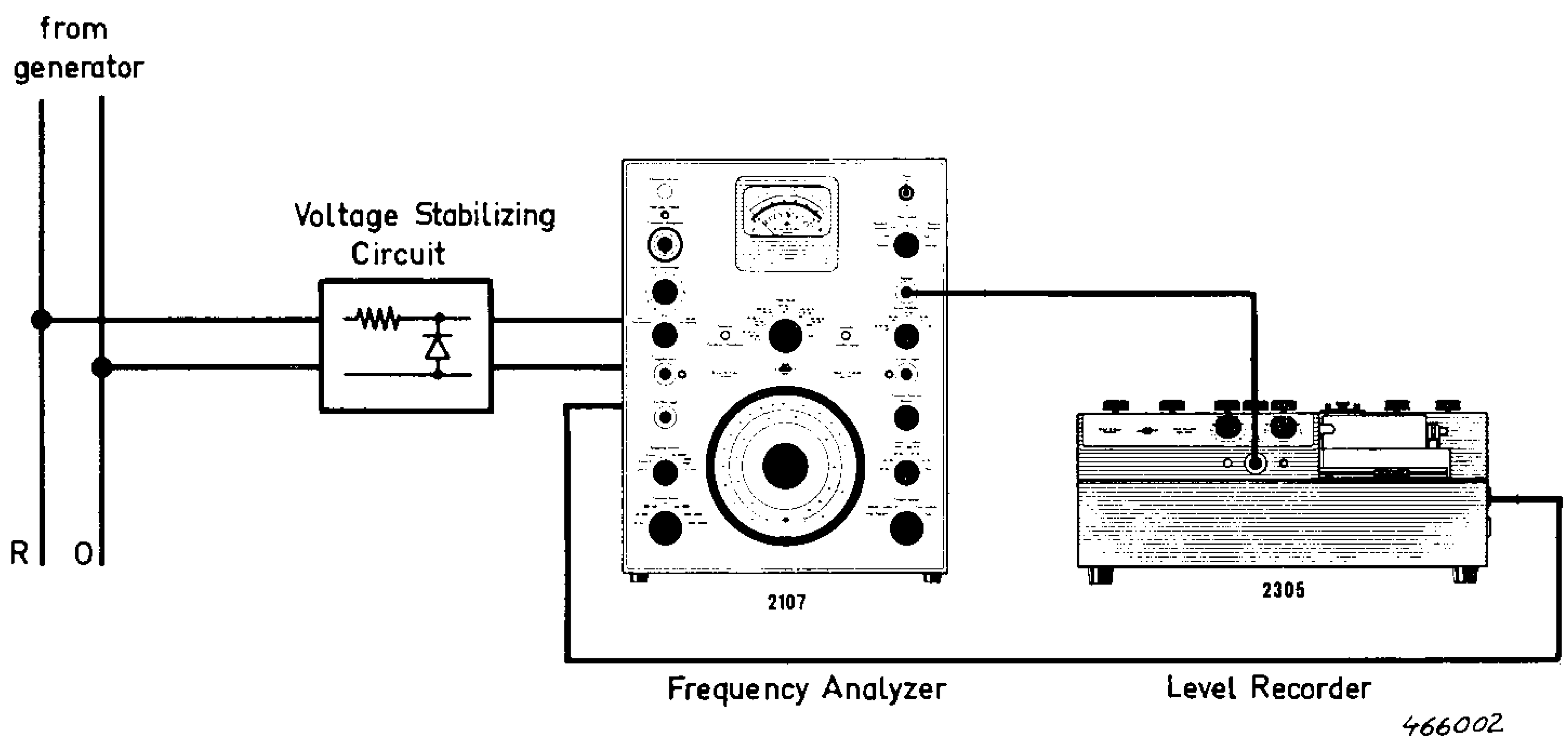

Fig. 1.

An interesting measuring arrangement for the automatic recording of changes in speed of rotation of AC generators can be made by means of the Brüel & Kjær Frequency Analyzer Type 2107 and Level Recorder Type 2305, see Fig. 1. As can be seen the arrangement is very simple and contains, apart from the above mentioned instruments, only a simple voltage stabilization circuit consisting of a transformer, a resistor and a Zener diode. The reason for including this stabilization arrangement will be clear from the discussion below.

As the speed of rotation of the generator is directly proportional to the frequency of the generator output voltage, the measurement consists basically in converting a change in frequency to a change in voltage, the voltage then being recorded automatically on the Level Recorder. To obtain the frequency-to-voltage conversion the Analyzer Type 2107 should be switched to its "40 dB Octave Selectivity Condition" and the normal operating point chosen on the steepest slope of the filterskirt, see Fig. 2. Here the normal operating frequency was 50 Hz and the Analyzer was tuned to 56 Hz.

Due to the logarithmic recording range of the Type 2305 Level Recorder the rather nonlinear scale for the frequency-to-voltage conversion is, when recorded, approaching "linearity" (see Fig. 2). However, when the generator frequency changes due to a change in loading the generator output **voltage** also changes. This is a very undesirable feature from the above described measuring point of view, but can be easily compensated for by the use of the previously described voltage stabilization circuit, see also Fig. 3. The stabilization circuit converts the original sinewave signal into a squarewave type signal

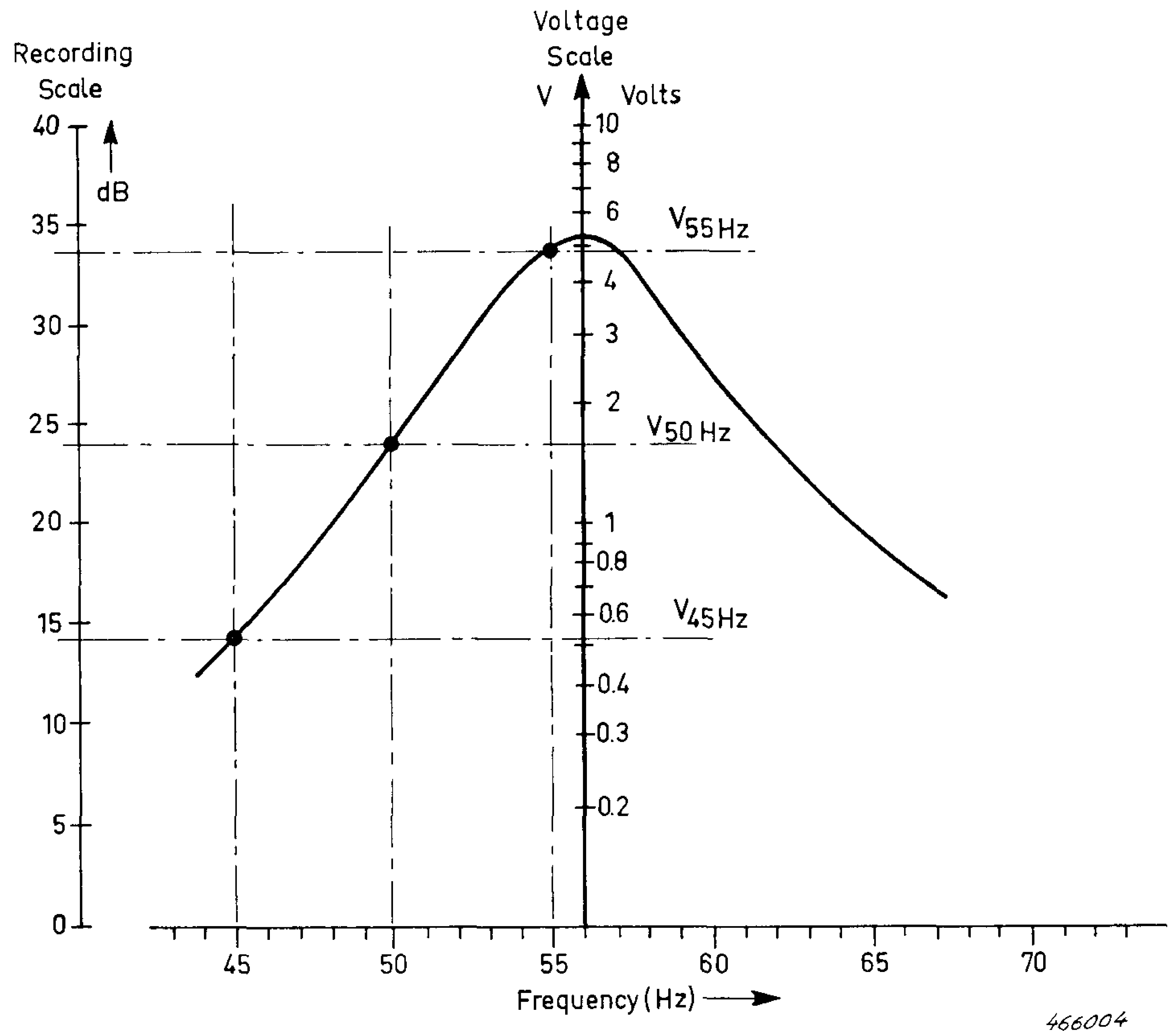

Fig. 2.

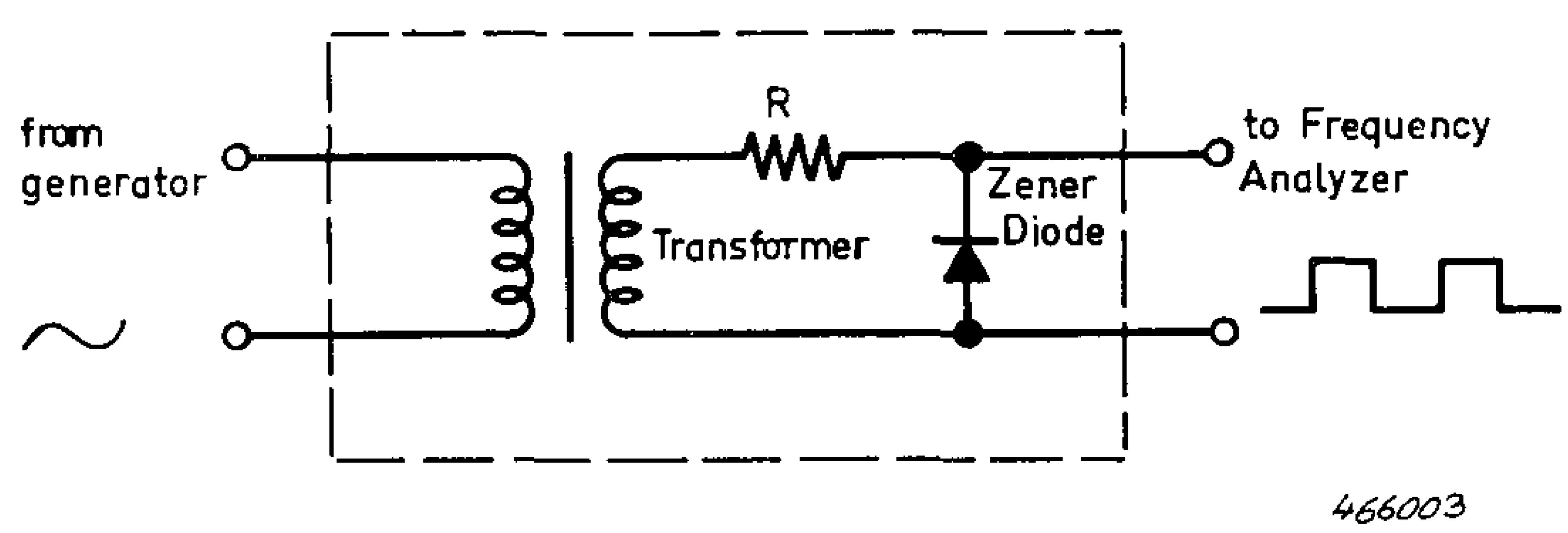

Fig. 3.

with constant amplitude before it is applied to the Analyzer. Even though the squarewave contains a number of harmonics this is unimportant in the case considered as these harmonics are filtered out by the Analyzer itself.
To calibrate the arrangement use should be made of a high quality frequency oscillator (and an electronic counter) and the voltage stabilizing arrangement should be kept in circuit.

AUG. 1966

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